Comparative Analysis of Some Social Choice Procedures

Sencer Ecer*

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Abstract

In this study the normative properties of various social choice rules are investigated as well as the rationality of the social decision they produce with respect to several rationality constraints. Arrovian rules, i.e., the rules that satisfy the locality condition (an analog of independence of irrelevant alternatives), are the main focus, but other common rules, such as Plurality and Borda, are also analyzed for practical purposes. The normative conditions and rationality constraints are the commonly-used ones in the literature. Whenever a counter example is provided, the most general form of the counter example is presented to shed light on the possible domain restrictions. The results are summarized in several tables. These tables can shed light on the choice of appropriate rule in a specific context.

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*Address: Department of Economics, University of Texas, Austin, TX 78712, E-mail: ecer@eco.utexas.edu, Phone: (512) 475 8551, Fax: (512) 471 3510. This paper was prepared for my master's thesis at the Boğaziçi University in 1996 under Fuad Aleskerov's supervision, however only the part related to Coalitional Pareto Rules was submitted due to the large volume of the study. I thank Fuad Aleskerov, Ahmet Alkan, Szilvia Papai, Murat Sertel and seminar participants at Boğaziçi University for suggestions. I also thank Borghan Narajabad who provided a counter example for one of the cases. Of course all errors are solely mine.

1 Introduction

In this paper I study several social choice rules. The focus of voting theory, the subject of the present paper, is well summarized in the introduction of Aizerman and Aleskerov's (1995) Theory of Choice.[1]

> 'Voting theory studies the following model: a group of any finite number n of individuals (voters) considers a finite set A of options (these can be candidates, plans, projects, etc.). Within the framework of some constraints that are similar for everybody[2] the voters can formulate their own opinions about the options to be selected. The problem is to 'process' the generally non-coinciding decisions of the voters into a single social decision meeting the same constraints, if any. Consideration can be given either to the deterministic or the probabilistic statement of the problem. Below, we discuss the deterministic one. Analysis of not only some social processes where n can be regarded as sufficiently great, but also of collective decision making in 'small groups' such as boards, committees, meetings, etc., reduces to the models of this kind.'

In Section 2, I introduce the necessary notation and various conventions to be used throughout the paper. In Section 3, I introduce the social choice models in the literature, that is, Social Choice Correspondences, Functional Voting Rules and Social Decision Rules. Then I give definitions of the 14 rules to be analyzed from different models, acoupled with illustrative examples. In Section 4, I list the definitions and intuitive explanations of various normative and rationality conditions (specific to each social choice model) introduced in the literature and provide a survey of results with regard to the relationships between these conditions. In Section 5, I analyze the rules first with respect to normative conditions then with respect to rationality constraints. In Section 6, I conclude by providing the results in Tables where I indicate whether a rule satisfies a certain condition. In the Appendix, I provide a list of formal definitions of 37 social choice rules.

2 Preliminaries

Let $A = \{a, b, ..., x, y, ...\}$ be a finite non-empty set of alternatives with $|A| \geq 2$, and a presentation X be any non-empty subset[3] of A, i.e., $\emptyset \neq X \subseteq A$. Let $N = \{1, 2, ..., n\}$ be a finite non-empty set of individuals (voters) with $|N| \geq 2$; a coalition ω be any non-empty subset of voters, i.e., $\emptyset \neq \omega \subseteq N$. Set of all presentations and set of all coalitions will be denoted by $\mathcal{A}$ and Ω, respectively, i.e., $\mathcal{A} = 2^A \backslash \{\emptyset\}$ and $\Omega = 2^N \backslash \{\emptyset\}$.

[1] Theory of Choice is the earliest book that presents the Western reader the advancements in this theory achieved in Russia during the post-war period.

[2] The constraints can be exemplified by pointing out the number of options to be selected, etc.

[3] Hereafter $\subset$ will dentoe the strict inclusion and $\subseteq$ will denote the weak inclusion.

A *Social Choice Procedure* (SCP) is a function that is used to aggregate individual opinions on a given presentation X in order to have a social decision on X. The forms of individual opinions and social decision will be explicitly defined for different models of SCPs in the next section. Unless otherwise stated the grand coalition, i.e., N, will be the given coalition and 'set of voters' or just 'voters' will mean N. To prepare reader for descriptions of SCPs some general notions are in order.

The opinions of voters might be in the form of a binary relation or a choice function and this is also true for a social opinion (decision). For the case of binary relations, weak orders[4] will be considered most generally as individual opinions. But sometimes, I will assume the given weak order to be a connected weak order, i.e., a linear order[5]. From now on $\mathcal{W}$ denote the set of all weak orders on A, $\mathcal{LO}$ will denote the set of all linear orders on A. Throughout this text, a weak order or linear order will be interpreted as a preferenceand sometimes 'a is preferred to b' may be used for aPb.

A *choice function* $C(\cdot)$ is defined as $C : \mathcal{A} \to 2^A$ under the condition $C(X) \subseteq X$ for any $X \in \mathcal{A}$. Let us denote as $\mathcal{C}$ the set of all choice functions defined on A.

To denote a weak order P (resp. choice function $C(\cdot)$) which is the opinion of individual $i \in N$ on A , P_i (resp. $C_i(\cdot)$) will be used. Given A and N, a profile $\overrightarrow{P}$ (resp. $\overrightarrow{C}(\cdot)$) of weak orders (resp. choice functions) is the set of individual opinions, i.e., $\overrightarrow{P} = \{P_i\}_{i \in N}$ (resp. $\overrightarrow{C} = \{C_i(\cdot)\}_{i \in N}$) and set of all profiles are denoted by $\mathcal{W}^N$ for weak orders and $\mathcal{C}^N$ for choice functions. To demonstrate a profile of weak orders the following convention will be used throughout the text.

P_1	P_2	P_3	P_4	P_5
a	b	a	c	a
d	a, d	b, c, d	b	b, c
b	c		a	d
c			d	

This 'table' represents a profile of weak orders belonging to a society consisting of 5 individuals. Since properties of a weak order enables one to establish either equivalency or preference of one to the other between any two alternatives, the above convention is able to represent a profile of weak orders, however for sake of clearance the following two remarks are made. 1) Any alternative is preferred to every alternative placed below and 2) the alternatives placed in the same line belong to an equivalence class, that is, none is preferred to neither of them. A similar convention is for profiles of

[4]A weak order is a binary relation with the following three properties: irreflexivity, transitivity and negative transitivity (i.e.,$\forall x, y, z \in A \quad xPz \Rightarrow xPy$ or yPz, where P is a binary reation on A). These properties enable one to establish either equivalency of two alternatives or dominance of one to the other.

[5]A linear order is a connected weak order where connectedness is defined as follows. $\forall x, y \in A$ $x \neq y \Rightarrow (xPy$ or $yPx)$. Properties of a linear order allow us to compare any two alternatives so that one can list the alternatives in order of preference, starting from the most preferred (top) element and ending with the least preferred (worst) element.

choice functions as follows:

$\underline{X}$	$\underline{C_1(\cdot)}$	$\underline{C_2(\cdot)}$	$\cdot$	$\underline{C_n(\cdot)}$	$\underline{C(\cdot)}$
A	$C_1(A)$	$C_2(A)$	$\cdot$	$C_n(A)$	$C(A)$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
X	$C_1(X)$	$C_2(X)$	$\cdot$	$C_n(X)$	$C(X)$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

The first column of the table can contain all the possible presentations however this is not always necessary for our purposes. All but last of the remaining columns are individual choice functions, that is the choice sets of the individual in response to the presentations, and the last column is the social decision.

For both versions of the convention the following statements are true. In this format, a point in an individual choice function or an individual preference such as $(\cdot)$ denotes that any of the existing possibilities are permitted. Such a point in social decision permits changes according to changes in parameters but this does not affect the point that is being made. Sometimes, numbers or parameters may stand in the column headings for individual preferences or choice functions which may either denote that so many individuals identically have such preferences or choice functions, or they may denote the indices of voters. For example, below there is a preference profile $\overrightarrow{P}$ and a profile of choice functions $\{C_i(\cdot)\}$. In $\overrightarrow{P}$, the convention allows us to know that 3 individuals have the first, $k-1$ individuals have the second and 1 individual have the third preference, respectively. Similarly, in $\{C_i(\cdot)\}$, 4 individuals have the first, k individuals have the second and 3 individuals have the third choice function, respectively. To denote that preferences in $\overrightarrow{P'}$ belong to 1^{st}, 2^{nd} and 3^{rd} voters, these indices are written in order above the preferences. Similar conventions can be used for choice functions.

	$\overrightarrow{P}$	
$\underline{3}$	$\underline{k-1}$	$\underline{1}$
b	a	c
c	b, c	$\cdot$
d	d	$\cdot$
a, e	e	d, e

$\underline{X}$	$\underline{4}$	$\underline{k}$	$\underline{3}$
$\{x, y, z\}$	$\{x, y\}$	$\{y\}$	$\{x\}$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\{y, z\}$	$\{y\}$	$\cdot$	$\{y\}$
$\cdot$	$\cdot$	$\cdot$	$\cdot$
$\cdot$	$\cdot$	$\cdot$	$\cdot$

	$\overrightarrow{P'}$	
$\underline{1}$	$\underline{2}$	$\underline{3}$
a	b, c	b
b	d	$\cdot$
d	a	$\cdot$
c, e	e	c, e

Sometimes it will be necessary to constrain the initial opinions of voters (in the form of weak orders) to $X \subset A$. Given P_i on A and $i \in N$, contraction of P_i to $X \subset A$ will be denoted by P_i/X, where $P_i/X = P_i \cap (X \times X)$ and contraction of a profile $\overrightarrow{P}$ will be denoted as $\overrightarrow{P}/X$ where $\overrightarrow{P}/X = \{P_i/X\}_{i\in N}$. Similarly, it may be necessary to contract the profile to a proper subset of N, such as $\omega \subset N$, then I use $\overrightarrow{P}/\omega$ where $\overrightarrow{P}/\omega = \{P_i\}_{i\in\omega}$, or I may need to contract the profile both to X and ω, in which case I use $\overrightarrow{P}/(X, \omega) = \{P_i/X\}_{i\in\omega}$.

Given $\overrightarrow{P} \in \mathcal{W}^N$ some SCPs require to define for each voter the set of alternatives preferred to the given alternative $x \in A$, i.e., the upper contour sets of $x \in A$ with respect to voters. The upper contour set of any $x \in A$ with respect to $i \in N$ is defined as $D_i(x) = \{y \in A : yP_ix\}$. Similarly, lower contour sets can be defined for any $x \in A$ and $i \in N$, i.e., $L_i(x) = \{y \in A : xP_iy\}$.

Given an alternative $a \in A$ and a profile of weak orders $\overrightarrow{P}$, some SCPs operate with the number of voters who do not prefer any other alternative $b \in A\backslash\{a\}$ to the given alternative, that is who declare a as best alternative or who give a a 'top' vote. This number will be denoted by $n^+(a, \overrightarrow{P})$, i.e. $n^+(a, \overrightarrow{P}) = |\{i \in N : |D_i(a)| = 0\}|$. This notation can be generalized as follows: $n^+_{j+1}(a, \overrightarrow{P}) = |\{i \in N : |D_i(a)| \leq j\}|$ where $j \in \{0, 1, ..., |A| - 1\}$. Then $n^+_1(a, \overrightarrow{P}) = n^+(a, \overrightarrow{P})$.

Given an alternative $x \in A$ and a profile of choice functions, $V_N(x, X; \{C_i(\cdot)\})$ (hereafter without subscript N, $V(x, X; \{C_i(\cdot)\})$ for short) will denote the set of voters who include x in their choice sets from X, i.e., $\{i \in N : x \in C_i(X)\}$. If the set of voters under consideration is different from the grand coalition such as $\omega_1 \subset N$, then this will be stated as a subscript, i.e., $V_{\omega_1}(x, X; \{C_i(\cdot)\})$. Similarly, given two alternatives $x, y \in A$ and a profile of binary relations, $V_N(x, y; \{P_i\})$ (hereafter $V(x, y; \{P_i\})$ for short) will denote the set of voters who include (x, y) in their preferences.

Given the grand coalition N, $\omega \in \Omega$ is a majority of N if $|\omega| \geq \lceil n/2 \rceil$, where $\lceil x \rceil$ denotes smallest integer greater than or equal to x. One can similarly define majority not only for N but also for any coalition $\omega \in \Omega$. Let $n(a, b)$ denote the number of voters who prefer a to b under $\overrightarrow{P}$, i.e., $n(a, b) = |\{i \in N : aP_ib\}|$. Using this, the *majority relation* μ is defined as follows:

$$a\mu b \Longleftrightarrow n(a, b) > n(b, a).$$

3 Social Choice Models and Rules

A SCP belongs to one of the following models defined with regard to the form individual opinions are given and social decision is to be constructed. It may aggregate individual binary relations either to a social binary relation or to a social choice function or it may aggregate individual choice functions to a social choice function. Then given A and N, it follows that there are three models for SCPs that are going to be examined:

(1) $F : \mathcal{A} \times \Omega \times \mathcal{W}^n \to \mathcal{C}$ (Social Choice Correspondences)

(2) $F : \mathcal{A} \times \Omega \times \mathcal{C}^n \to \mathcal{C}$ (Functional Voting Rules)

(3) $F : \mathcal{A} \times \Omega \times \mathcal{W}^n \to \mathcal{B}$ (Social Decision Rules)

where F is the SCP, $\mathcal{B}$ is the set of all binary relations on A.

Remark 1: As stated before, the set of all weak orders will be considered as the most general form in which individual opinions are given, but the social decision can be any kind of binary relation depending on the profile and the procedure.

The procedures that belong to the first model are called as 'Social Choice Correspondences (SCCs)' and respectively those in the second model 'Functional Voting Rules (FVRs)' and in the third, 'Social Decision Rules (SDRs)'.

Following examples are to illustrate the three models. Throughout the examples, whole set of alternatives is the given presentation hence $X = A = \{a, b, c, d\} = X$ and the grand coalition is the given coalition hence $\omega = N = \{1, 2, 3, 4, 5\}$.

Example 1 (SCC) *Let $\overrightarrow{P} \in \mathcal{W}^n$ be as follows.*

P_1	P_2	P_3	P_4	P_5
a	b	a	c	a
d	a, d	b, c	b, d	b, c
b	c	d	a	d
c				

Let $F : \mathcal{A}\times\Omega\times\mathcal{W}^n \to \mathcal{C}$ be defined as follows. Given $\overrightarrow{P} \in \mathcal{W}^n$, the social decision on $X \in \mathcal{A}$ is the value that a choice function $C(\cdot) \in \mathcal{C}$ determines by choosing the element(s) in X which is declared best by at least a majority[6] of the given $\omega \in \Omega$, i.e., $F(\overrightarrow{P}, X, \omega) = F(\overrightarrow{P}, A, N) = F(\overrightarrow{P}) = \{x \in A : n^+(x, \overrightarrow{P}) \geq \lceil n/2 \rceil\}$. From the above profile it can be seen that, $n^+(a, \overrightarrow{P}, X, \omega) = n^+(a, \overrightarrow{P}, A, N) = n^+(a, \overrightarrow{P}) = 3$ and $n^+(b, \overrightarrow{P}) = n^+(c, \overrightarrow{P}) = 1$ and $n^+(d, \overrightarrow{P}) = 0$. Since $n^+(a, \overrightarrow{P}) = 3 > \lceil n/2 \rceil = 2$ and $\rceil \exists x \in A \backslash \{a\}$ such that , $n^+(x, \overrightarrow{P}) \geq \lceil n/2 \rceil$, $F(\overrightarrow{P}) = C(A) = \{a\}$.

Example 2 (FVR) *(FVR) Let $\overrightarrow{C} \in \mathcal{C}^n$ have the values on A as follows:*

X	$C_1(\cdot)$	$C_2(\cdot)$	$C_3(\cdot)$	$C_4(\cdot)$
A	$\{b, c\}$	$\{a\}$	$\{c\}$	$\{c\}$

Let $F : \mathcal{A} \times \Omega \times \mathcal{C}^n \to \mathcal{C}$ be defined as follows. Given $\overrightarrow{C}$ the social decision $F(\overrightarrow{C})$ is determined by a choice function $C(\cdot) \in \mathcal{C}$ which chooses those elements from the given presentation X that maximal number of voters[7] (from the given coalition) include in their choice sets, i.e., $F(\overrightarrow{C}, X, \omega) = F(\overrightarrow{C}, A, N) = F(\overrightarrow{C}) = \{x \in X : \forall y \in A \quad |V(x, A; \{C_i(\cdot)\})| \geq |V(y, A; \{C_i(\cdot)\})|\}$. From the above profile it can be seen that, $|V(a, A; \{C_i(\cdot)\})| = |V(b, A; \{C_i(\cdot)\})| = 1$ and $|V(c, A; \{C_i(\cdot)\})| = 3$ and $|V(d, A; \{C_i(\cdot)\})| = 0$. So social decision on A is $F(\overrightarrow{C}) = C(A) = \{c\}$.

Example 3 (SDR) *Let $\overrightarrow{P} \in \mathcal{W}^n$ be as in Example* **??***.*

Let $F : \mathcal{A} \times \Omega \times \mathcal{W}^N \to \mathcal{B}$ be defined as follows. Given $\overrightarrow{P}$, the social decision is a binary relation $P \in \mathcal{B}$ where $F(\overrightarrow{P}) = P = \{(x, y) \in X \times X : |V(x, y; \{P_i\})| \geq \lceil n/2 \rceil\}$.

[6]The SCC being described in the example is the well-known 'Simple Majority' rule.

[7]The SCP being described is 'Approval Voting'.

Since $\lceil n/2 \rceil = \lceil 5/2 \rceil = 3$, the social decision $P = \{(a,b),(a,c),(a,d),(b,d)(c,d)\}$ or

$$\begin{array}{c} \underline{P} \\ a \\ b, c \\ d \end{array}$$

Social decision P is a weak order in this example but this will not necessarily be the case as will be seen.

3.1 Social Choice Correspondences

In this section, the individual opinions are represented as weak orders and the social decision is constructed as a choice function. The Social Choice Correspondences studied here are classified as follows:

1. Coalitional Pareto Rules,
2. Positional SCCs.

Throughout the following definitions the expressions are to hold $\forall \overrightarrow{P} \in \mathcal{W}^n$, $\forall \omega \in \Omega$, $\forall X \in \mathcal{A}$.

3.1.1 Coalitional Pareto Rules (Aleskerov, 1992)

In this subsection, given A ,N and $\overrightarrow{P} \in \mathcal{W}^n$ social decision on presentation X, i.e., $F(\overrightarrow{P}/X)$ will be denoted as $C(X)$. Throughout the definitions $\mathcal{I}$ will denote the set of all coalitions with cardinality greater than or equal to k, i.e., $\mathcal{I} = \{I \subseteq N : |I| = k\}$ where $1 \leq k \leq n$ and k is a positive integer.

1 *Strong k-majoritarian q-Pareto rule*

Let $f(\overrightarrow{P}/(\{i\},X);q)$ be defined as q top alternatives in $\overrightarrow{P}$ when contracted to coalition $\{i\}$ and presentation X, that is q top alternatives[8] in P_i. The rule chooses an alternative if it is among the top q alternatives in the preference of every agent in some $I \in \mathcal{I}$ where $\mathcal{I}$ is constructed for a given k, i.e.,

$$C(X) = \bigcup_{I \in \mathcal{I}} \bigcap_{i \in I} f(\overrightarrow{P}/(\{i\},X);q),$$

where $f(\overrightarrow{P}/(\{i\},X);q) = \{x \in X : |X \cap D_i(x)| \leq q\}$.

This rule can also be defined by using a different notion, that is, for each alternative counting the number the voters who place it among top q and checking if this number equals or exceeds k, i.e.,

$$C(X) = \{x \in X : n^+_{q+1}(x, \overrightarrow{P}/X) \geq k\}.$$

[8]For the present model, it will be assumed that $0 \leq q \leq |A| - 1$. This is because $q < 0$ implies that $\forall X \quad C(X) = \emptyset$ and $q \geq |A|$ implies that $\forall X \quad C(X) = X$.

Example 4 *Consider the following example.*

P_1	P_2	P_3	P_4	P_5
a	b	d	b	a
d	a, d	a, b, c	c	b, c
b	c		a	d
c			d	

Let $k = 2$ and $q = 1$. Hence $\mathcal{I} = \{I \subseteq N : |I| \geq 2\}$. Now consider $I' = \{1, 2\} \in \mathcal{I}$. Since $f(X, \overrightarrow{P}; \{1\}, 1) \cap f(X, \overrightarrow{P}; \{2\}, 1) = \{a, d\} \cap \{a, b, d\} = \{a, d\}$, $\{a, d\} \subseteq C(A)$. Now consider $I'' = \{3, 5\} \in \mathcal{I}$. Since $f(X, \overrightarrow{P}; \{3\}, 1) \cap f(X, \overrightarrow{P}; \{5\}, 1) = A \cap \{a, b, c\} = \{a, b, c\}$, $\{a, b, c\} \subseteq C(A)$. Thus it is unnecessary to check the other coalitions in $\mathcal{I}$, hence $C(A) = A$.

2 Strongest *k-majoritarian q-Pareto rule*

Let $f(\overrightarrow{P}/(I, X); q)$ be defined as alternatives in profile $\overrightarrow{P}$ contracted to coalition I and presentation X, having upper contour sets having an intersection with cardinality smaller than or equal to q. Another way of saying this is, $f(\overrightarrow{P}/(I, X); q)$ is the set of alternatives that are q-Pareto Optimal under $\overrightarrow{P}$ with repect to I and X. The rule chooses an alternative if it is q-Pareto Optimal under $\overrightarrow{P}$ with repect to all the coalitions $I \in \mathcal{I}$, i.e.,

$$C(X) = \bigcap_{I \in \mathcal{I}} f(\overrightarrow{P}/(I, X); q),$$

where $f(X, \overrightarrow{P}; I, q) = \{x \in X : \left|\bigcap_{i \in I} X \cap D_i(x)\right| \leq q\}$.

Example 5 *Consider the following example.*

P_1	P_2	P_3	P_4	P_5
a	b	d	b	a
d	a, d	a, b	a	b
b	c	c	c, d	c, d
c				

Let $k = 3$ and $q = 0$. Hence $\mathcal{I} = \{I \subseteq N : |I| \geq 3\}$. Now consider $I' = \{1, 2, 3\} \in \mathcal{I}$ and $c \in A$. Since $\left|\bigcap_{i \in I'} A \cap D_i(c)\right| = |\{a, b, d\}| = 3 > q = 0$ that is $f(A, \overrightarrow{P}; I', 0) = \{a, b, d\}$, hence $c \notin C(A)$. Then consider $d \in A$ and $I'' = \{1, 4, 5\}$. Since $\left|\bigcap_{i \in I''} A \cap D_i(d)\right| = |\{a\}| = 1 > q = 0$, $d \notin C(A)$. Then consider $a \in A$. Since $\forall i \in \{1, 5\}$ $|A \cap D_i(a)| = 0$, if $1 \in I$ or $5 \in I$ then $\left|\bigcap_{i \in I} A \cap D_i(a)\right| = 0 \leq q = 0$ therefore consider only $I''' = \{2,3,4\}$. Since $\left|\bigcap_{i\in I'''} A \cap D_i(a)\right| = |\{b\}\cap\{d\}\cap\{b\}| = 0 \le q = 0$, $a \in C(A)$. Finally, consider $b \in A$. Since $\forall i \in \{2,4\} \quad A \cap D_i(b) = 0$, if $2 \in I$ or $4 \in I$ then $\left|\bigcap_{i\in I} A \cap D_i(b)\right| = 0 \le q = 0$ therefore consider only $I'^v = \{1,3,5\}$. Since $\left|\bigcap_{i\in I'^v} A \cap D_i(b)\right| = |\{a,d\}\cap\{d\}\cap\{a\}| = 0 \le q = 0$, $b \in C(A)$. Hence $C(A) = \{a,b\}$.

3 *Weak k-majoritarian q-Pareto rule*

Let $f(\overrightarrow{P}/(I,X);q)$ be, as before, the set of alternatives that are q-Pareto Optimal under $\overrightarrow{P}$ with repect to I and X. The rule chooses an alternative if it is q-Pareto Optimal under $\overrightarrow{P}$ with repect to at least one coalition $I \in \mathcal{I}$, i.e.,

$$C(X) = \bigcup_{I\in\mathcal{I}} f(\overrightarrow{P}/(I,X);q),$$

where $f(X,\overrightarrow{P};I,q) = \{x \in X : \left|\bigcap_{i\in I} X \cap D_i(x)\right| \le q\}$.

Example 6 *Consider the following example.*

P_1	P_2	P_3	P_4	P_5
a	b	d	b	a
d	a,d	a,b	a	b
b	c	c	c,d	c,d
c				

Let $k=3$ and $q=0$. Hence $\mathcal{I} = \{I \subseteq N : |I| \ge 3\}$. Now consider $I' = \{1,2,3\} \in \mathcal{I}$. Since $f(X,\overrightarrow{P};I',0) = \{a,b,d\}$, $\{a,b,d\} \subseteq C(A)$. Now consider $c \in A$. Since $\forall i \in N \quad A \cap D_i(c) \ge 2$, $\forall I \in \mathcal{I} \quad \left|\bigcap_{i\in I} X \cap D_i(x)\right| \ge 2 > q = 0$ therefore $c \notin C(A)$ hence $C(A) = \{a,b,d\}$.

Note that in each of CPRs, there is some 'penalty' function which assigns some numerical values to alternatives. These values represent the undesirability of the alternative to society and society decides on the alternatives by choosing the one(s) having penalties below the critical value q.

3.1.2 Positional SCCs

For expository purposes, the given presentation will be the set of all alternatives A in the definitions and hence $C(A) = F(\overrightarrow{P})$.

4 *Plurality Rule*

The alternative declared the best by a maximal number of voters is chosen, i.e.,

$$C(A) = \{a \in A : \forall x \in A \quad n^+(a, \overrightarrow{P}) \geq n^+(x, \overrightarrow{P})\}.$$

Example 7 *Consider the following example.*

P_1	P_2	P_3	P_4	P_5
a	b	d	b	a
d	a, d	a, b, c	c	b, c
b	c		a	d
c			d	

In this example, $\forall x \in A \quad n^+(a, \overrightarrow{P}) = n^+(b, \overrightarrow{P}) = 2 \geq n^+(x, \overrightarrow{P})$ hence $C(A) = \{a, b\}$.

5 *Inverse Plurality*

The alternative declared worst by a minimal number of voters is chosen, i.e.,

$$a \in F(\overrightarrow{P}) \Longleftrightarrow [\forall x \in A, \quad n^-(a, \overrightarrow{P}) \leq n^-(x, \overrightarrow{P})].$$

(Example omitted).

6 *Borda Procedure*

Given a profile of linear orders $\overrightarrow{P}$, consider $x \in A$ and let each voter assign a score $r_i(x, \overrightarrow{P})$ to x which is the cardinality of lower contour set under $P_i \in \overrightarrow{P}$, i.e., $r_i(x, \overrightarrow{P}) = |L_i(x)| = |\{b \in A : aP_ib\}|$. The sum of these scores through every voter $i \in N$ is called the Borda Count of the alternative. Then choose the one who has highest Borda Count, i.e.,

$$C(A) = \{a \in A : \forall x \in A \quad r(a, \overrightarrow{P}) \geq r(x, \overrightarrow{P})\},$$

where $r(a, \overrightarrow{P}) = \sum_{i=1}^{n} r_i(a, P_i)$ and $r_i(a, P_i) = |L_i(x)|$.

Example 8 *Consider the following example.*

P_1	P_2	P_3	P_4	P_5
a	b	d	b	a
d	a, d	a, b, c	c	b, c
b	c		a	d
c			d	

In this example $\forall x \in A \quad r(a, \overrightarrow{P}) = r(b, \overrightarrow{P}) = 8 \geq r(x, \overrightarrow{P})$ hence $C(A) = \{a, b\}$.

7 *Inverse Borda Procedure*

Compute the Borda Count of each alternative as defined in Borda Procedure. Then eliminate the one who has lowest Borda Count and contract $\overrightarrow{P}$ to the remaininig set and compute the new Borda Scores in the contracted profile. Then eliminate another one similarly and go on like this until there is no alternative to eliminate from the contracted set, i.e.,

Let $r(a, \overrightarrow{P})$ be the Borda Count of any $a \in A$ under $\overrightarrow{P}$.

Then eliminate $c \in A$ where $\forall x \in A, \quad r(c, \overrightarrow{P}) \leq r(x, \overrightarrow{P})$.

and apply the same procedure to $X = A \backslash \{c\}$ and $\overrightarrow{P}/X$.

Continue with the procedure by contracting the set in consideration until reaching a social decision.

(Example omitted).

3.2 Functional Voting Rules

For expository purposes, the given presentation will be the set of all alternatives A in the definitions and hence $C(A) = F(\overrightarrow{C})$.

8 *Approval Voting*[9]

Given a profile $\overrightarrow{C} \in \mathcal{C}^n$, every $i \in N$ chooses $C_i(A)$ from A where $C_i(A) \subseteq A$. Then for each alternative $x \in A$, the number of voters who choose x from A is computed. The alternative with greatest such number is chosen if it is chosen at least by one voter, i.e.,

$$C(A) = \{x \in \bigcup_{i \in N} C_i(A) : \forall y \in A \quad \left|V(x, A; \overrightarrow{C})\right| \geq \left|V(y, A; \overrightarrow{C})\right|\}.$$

Example 9 *Consider the following example where* $A = \{a, b, c\}$.

X	$C_1(\cdot)$	$C_2(\cdot)$	$C_3(\cdot)$	$C_4(\cdot)$	$C_5(\cdot)$	$C(\cdot)$
A	$\{b\}$	$\{b\}$	$\{a, b\}$	$\{a\}$	$\{a, c\}$	$\{a, b\}$

In this example, $\left|V(a, A; \overrightarrow{C})\right| = \left|V(b, A; \overrightarrow{C})\right| = 3$ and $\left|V(c, A; \overrightarrow{C}\})\right| = 1$ hence $C(A) = \{a, b\}$.

[9]This version of Approval Voting is by (Sertel, 1988). Here, an alternative is chosen as social decision, only if it is chosen by at least one of the voters whereas in Fishburn and Brams' version whole presentation is chosen if each of the voters declare empty choice.

9 *k-majority rules* (Aizerman and Aleskerov, 1986)

Given $\overrightarrow{C}(\cdot)$ every $i \in N$ chooses $C_i(A)$ from A where $C_i(A) \subseteq A$. For each alternative the number of voters who indicated that alternative in his/her choice set is computed. The alternative is in the social decision $C(A)$ if there exists at least k voters who includes it in his/her choice set where $1 \leq k \leq n$, i.e.,

$$C(A) = \{x \in A : |V(x, A; \{C_i(\cdot)\})| \geq k\}.$$

Example 10 *Consider the following example where $A = \{a, b, c, d\}$.*

X	$C_1(\cdot)$	$C_2(\cdot)$	$C_3(\cdot)$	$C_4(\cdot)$	$C_5(\cdot)$	$C_6(\cdot)$
A	$\{b\}$	$\{b\}$	$\{a\}$	$\{a\}$	$\{c, a\}$	$\{d\}$

Let $k = 3 < n = 6$. In this example, $\forall x \in A \backslash \{a\} \quad |V(x, A; \{C_i(\cdot)\})| < k = 3$ and $|V(x, A; \{C_i(\cdot)\})| = 3 \geq k = 3$ hence $C(A) = \{a\}$.

10 *Voting with Veto* (Aleskerov, forthcoming)

In this procedure, there are a number of individuals forming a non-empty coalition ω_0 that are called vetoers. They are less than the majority of N. For an alternative to be chosen a majority of N including all the vetoers should choose it. So to be chosen by all the vetoers is not enough for an alternative to be socially chosen but not to be chosen by only one of the vetoers is enough for it to be out of social decision, i.e.,

$x \in C(A) = F(\overrightarrow{C}) \Leftrightarrow$ (1) and (2) where (1) $x \in \bigcap\limits_{i \in \omega_0} C_i(A)$ and

(2) $|V_N(x, A; \{C_i(\cdot)\})| \geq \lceil n/2 \rceil$.

Example 11 *Consider the following example where $A = \{a, b, c\}$.*

X	$C_1(\cdot)$	$C_2(\cdot)$	$C_3(\cdot)$	$C_4(\cdot)$	$C_5(\cdot)$	$C(\cdot)$
A	$\{b, c\}$	$\{a, b\}$	$\{a, c\}$	$\{a, c\}$	$\{b\}$	$\{b\}$

Let $\omega_0 = \{1, 2\}$. Since $a \notin C_1(A)$ and $c \notin C_2(A)$ and $b \in C_1(A) \cap C_2(A)$ and $|V_N(b, A; \{C_i(\cdot)\})| = 3 \geq \lceil 5/2 \rceil = 3$, $C(A) = \{b\}$.

3.3 Social Decision Rules

For expository purposes, the given presentation will be the set of all alternatives A in the definitions and hence $P = F(\overrightarrow{P})$.

Given a profile of linear orders $\overrightarrow{P} \in \mathcal{LO}^n$, first define distance matrices R_i corresponding to each voter $i \in N$. Then the rule will be defined in terms of distances computed from these matrices.

Definition 1 $\forall a, b \in A \quad R_i = \{r^i_{ab}\}$ *and* $r^i_{ab} = 1$ *if* aP_ib, $r^i_{ab} = 0$ *otherwise.*

Definition 2 *Let* $d(P_i, P_j)$ *denote the distance of a linear order to another linear order. Then* $\forall a, b \in A \quad \forall i, j \in N \quad d(P_i, P_j) = \sum\limits_{a,b} \left| r^i_{ab} - r^j_{ab} \right|$. *The distance* $d(P_i, \overrightarrow{P})$ *of a linear order* $P_i \in \overrightarrow{P}$ *to the profile* $\overrightarrow{P} \in \mathcal{LO}^n$*, is defined as follows:* $d(P_i, \overrightarrow{P}) = \sum\limits_{j \in N} d(P_i, P_j)$.

In Kemeny rule the preference (linear order) with minimum distance to its profile is the social decision from that profile, i.e.,

$$P = P_k \text{ such that } \forall P_j \in \overrightarrow{P} \quad d(P_k, \overrightarrow{P}) \leq d(P_j, \overrightarrow{P}).$$

Example 12 *Consider the following example where* $A = \{x, y, z, w\}$ *and* $N = \{1, 2, 3\}$*. For simplicity corresponding distance matrices are added.*

$\overrightarrow{P}$

P_1	P_2	P_3
x, y	x, y	x
z	z, w	y, z, w
w		

R_1

–	x	y	z	w
x	–	0	1	1
y	0	–	1	1
z	0	0	–	1
w	0	0	0	–

R_2

–	x	y	z	w
x	–	0	1	1
y	0	–	1	1
z	0	0	–	0
w	0	0	0	–

R_3

–	x	y	z	w
x	–	1	1	1
y	0	–	0	0
z	0	0	–	0
w	0	0	0	–

Since $\sum\limits_{k \in N} d(P_2, P_k) = 4 < \sum\limits_{k \in N} d(P_2, P_k) = 5 < \sum\limits_{k \in N} d(P_3, P_k) = 7,\ P_3 \neq P = P_1$.

12 (k_1, k_2)-*majority* (Aleskerov and Vladimirov, 1986)

Given a profile of weak orders $\overrightarrow{P}$ and N , in this procedure the pair $(x, y) \in A \times A$ is included in social decision P if number of voters who include this pair in their preferences is at least k_1 and number of voters who does not include this pair in their preferences is at most k_2 where $1 \leq k_1 \leq n$ and $0 \leq k_2 \leq n$, i.e.,

$$P = \left\{(x, y) \in A \times A : \left|V(x, y; \overrightarrow{P})\right| \geq k_1 \text{and } \left|V(x, y; \overrightarrow{P})\right| \leq k_2.\right\}$$

(Example omitted).

13 *Absolute k-majority*

This procedure is a (k_1, k_2) procedure where $k_1 + k_2 = n$. Given a profile of weak orders $\overrightarrow{P}$ and N , in this procedure the pair $(x, y) \in A \times A$ is included in social decision P if number of voters who include this pair in their preferences is at least k independent of other preferences where $1 \leq k \leq n$, i.e.,

$$P = \left\{(x, y) \in A \times A : \left|V(x, y; \overrightarrow{P})\right| \geq k\right\}.$$

Example 13 *Consider the following example where* $A = \{x, y, z\}$.

$k-2$	1	1	$n-k$
x	y	y	x, y, z
y	x, z	z	
z		x	

In this example, $|V(y, z; \{P_i\})| \geq k$ and $\rceil\exists(a, b) \in A \times A\backslash(x, y)$ such that $|V(a, b; \{P_i\})| \geq k$, hence $P = \{(y, z)\}$.

14 *Relative k-majority*

This procedure is a (k_1, k_2) procedure where $k_2 = 0$. Given a profile of weak orders $\overrightarrow{P}$ and N , in this procedure the pair $(x, y) \in A \times A$ is included in social decision P if if number of voters who include this pair in their preferences is at least k and other voters abstain to include (x, y) or (y, x) in their preferences, i.e.,

$$P = \{(x, y) : |V(x, y; \{P_i\})| \geq k \text{ and}$$

$$card\{j \in N : (y, x) \notin P_j \text{ and } (x, y) \notin P_j\} = n - k\}.$$

Example 14 *Consider the following example where $A = \{x, y, z\}$.*

k	$n-k$
z	x, y
x	z
y	

In this example, $|V(x, y; \{P_i\})| \geq k$ and $card\{j \in N : (y, x) \notin P_j$ and $(x, y) \notin P_j\} = n - k\}$ and $\rceil \exists (a, b) \in A \times A \backslash (x, y)$ such that $|V(a, b; \{P_i\})| \geq k$ and $card\{j \in N : (b, a) \notin P_j$ and $(a, b) \notin P_j\} = n - k\}$, hence $P = \{(y, z)\}$.

4 Normative and Rationality Constraints

4.1 Normative Conditions

This section will introduce normative conditions (NCs) for Social Choice Procedures (SCPs) and some related theorems. NCs define subsets in the space of all SCPs. In the first three subsections various NCs will be defined for each of the choice models introduced in the previous chapter. Depending on the choice model, corresponding conditions will be introduced. Since the ideas behind NCs do not differ in spirit in any choice model the interpretations will only be given for NCs for Social Choice Correspondences (SCCs). In most cases normative conditions are 'natural' properties that a SCP should satisfy such as 'the value of each voter is the same', i.e., anonymity.

In the next three subsections NCs for SCCs, SDRs and FVRs will be introduced respectively. Generally an NC has three versions for each of the social choice models, which all have the same spirit therefore the explanations will be made when the condition is introduced the first time.

4.1.1 Normative Conditions for Social Choice Correspondences

In what follows, definitions of normative conditions for SCCs are introduced. From now on the abbreviations given in parentheses are going to be used for the names of the conditions. For the SCCs that satisfy the condition Q, Λ^Q will be used such as Λ^M for classes of rules that satisfy Monotonicity. Since the social decision here is a choice function, only $C(X)$ will be used instead of $F(\overrightarrow{P}/X)$. For definitions of NCs for SCCs every statement is true $\forall \overrightarrow{P}, \overrightarrow{P'} \in \mathcal{W}^n$ and $\forall x \in X$, $\forall X \in \mathcal{A}$, such that $x \in X$ as necessary. (When an addition is necessary this will be done explicitly).

Below the conditions 1-12 are from (Aleskerov, 1992) except Neutrality1 which is the usual Neutrality to alternatives of May (1952), and the conditions 13 and 14 are from (Moulin, 1989).

1) *Locality* (**L**)

In general Locality condition carries the spirit of Arrow 's Independence of Irrelevant Alternatives. Here an alternative will not lose its position in the social decision as long as its upper contour sets with respect to all the voters stay the same.

A rule F satisfies Locality condition if,

$$[\forall i \in N \quad X \cap D_i(x) = X \cap D'_i(x)] \Rightarrow [x \in C(X) \Leftrightarrow x \in C'(X)].$$

2) *Positive non-imposedness* (**NI**$^+$)

Positive non-imposedness condition requires that no alternative in no presentation can be 'banned' or 'vetoed' in a predetermined manner; so long as individuals want this alternative (and declare their preferences accordingly) it is possible for the alternative to be chosen.

A rule F satisfies Positive non-imposedness condition if,

$$\exists \overrightarrow{P} \in \mathcal{W}^n \text{ such that } x \in C(X).$$

3) *Negative non-imposedness* (**NI**$^-$)

Negative non-imposedness requires that no alternative in no presentation can be put in social decision in a predetermined manner; so long as individuals do not want this alternative (and declare their preferences accordingly) it is possible for the alternative not to be chosen.

A rule F satisfies Negative non-imposedness condition if,

$$\exists \overrightarrow{P} \in \mathcal{W}^n \text{ such that } x \notin C(X).$$

Remark 2: All the Social Choice Correspondences that I will consider satisfy **NI**$^+$ and **NI**$^-$ which can be easily checked by the reader.

4) *Monotonicity* (**M**)

Monotonicity is a strengthening of Locality condition. It requires the following. Assume that an alternative is chosen from a presentation under a certain profile. Under any other profile, as long as the position of the alternative do not happen to be worse for any body, that is, for no individual preference the upper contour set of the alternative became larger, the same alternative must be chosen from the same presentation.

A rule F satisfies Monotonicity condition if,

$$[\forall i \in N \quad X \cap D'_i(x) \subseteq X \cap D_i(x)] \Rightarrow [x \in C(X) \Rightarrow x \in C'(X)].$$

5) *Neutrality1* ($\mathbf{Ne_1}$)

Neutrality1 is the neutral treatment of the alternatives by the procedures. Then no alternative is different from the other in terms of the name it has.

Let $\sigma : A \to A$ be a bijection and let $\sigma(X) = \{\sigma(x) \in A : x \in X\}$. A rule F satisifes Neutrality1 if,

$$[\forall i \in N \quad \sigma(L_i(x)) = L'_i(\sigma(x)) \text{ and } \sigma(D_i(x)) = D'_i(\sigma(x))]$$

$$\Rightarrow [C'(\sigma(X)) = \sigma(C(X))].$$

Remark 3: All the Social Correspondences that I will consider satisfy $\mathbf{Ne_1}$ which can be easily checked by the reader.

6) *Neutrality2* ($\mathbf{Ne_2}$)

Neutrality2 is a strengthening of Locality condition like Monotonicity. It requires the rule to be independent of the name of the alternative and the presentation to which it belongs.

A rule F satisifes Neutrality2 if,

$$[\forall i \in N \quad D_i(x) \cap X = D_i(y) \cap X'] \Rightarrow [x \in C(X) \Leftrightarrow y \in C(X')],$$

and for any bijection $\sigma : A \to A$,

$$[\forall i \in N \quad \sigma(X \cap D_i(x)) = \sigma(X) \cap D_i(\sigma(x))] \Rightarrow$$

$$[x \in C(X) \Leftrightarrow \sigma(X) \in C(\sigma(X))].$$

Now the following Theorem can be introduced.

Theorem 1 *(Aleskerov, 1992)* $\Lambda^{\mathbf{Ne_2}} \subset \Lambda^{\mathbf{L}}$ *and* $\Lambda^{\mathbf{M}} \subset \Lambda^{\mathbf{L}}$.

7) *Anonymity* (**An**)

Anonymity condition requires the voters to be equally treated by the rule. Then no voter is different from the other in terms of the name it has.

A rule F satisifes anonymity if where $\eta : N \to N$ is a bijection and $\{P_{\eta(i)}\}_{\eta(i) \in N} = \overrightarrow{P'}$ one always has that $C(X) = C'(X)$.

Remark 4: All the Social Choice Correspondences that I will consider satisfy **An** which can be easily checked by the reader.

Definition 3 *The class of rules that satisifes* $\mathbf{NI^+}, \mathbf{NI^-}, \mathbf{M}$ *and* $\mathbf{Ne_2}$ *are called the Central Class denoted by* $\Lambda^{\mathbf{C}}$, *i.e.,* $\Lambda^{\mathbf{NI^+}} \cap \Lambda^{\mathbf{NI^-}} \cap \Lambda^{\mathbf{M}} \cap \Lambda^{\mathbf{Ne_2}} = \Lambda^{\mathbf{C}}$. *The rules from Central Class that satisfy* **An** *are called the Symmetrically Central Class denoted by* $\Lambda^{\mathbf{SC}}$, *i.e.,* $\Lambda^{\mathbf{C}} \cap \Lambda^{\mathbf{An}} = \Lambda^{\mathbf{SC}}$.

8) *Positive non-dominance* $(\mathbf{ND^+})$

Positive non-dominance condition requires that, given a profile if an alternative is top in at least one individual's preference then it must be chosen.

A rule F satisfies Positive non-dominance condition if,

$$[\exists i \in N \text{ such that } D_i(x) \cap X = \emptyset] \Rightarrow x \in C(X).$$

9) *Negative non-dominance* $(\mathbf{ND^-})$

Negative non-dominance condition requires that, given a profile if an alternative is not top in even one individual's preference then it must not be chosen.

A rule F satisfies Negative non-dominance condition if,

$$[\rceil \exists i \in N \text{ such that } D_i(x) \cap X = \emptyset] \Rightarrow x \notin C(X).$$

10) *Unanimity* $(\mathbf{U})$

Unanimity condition requires that, given a profile if an alternative is top in everybody 's preference then it must be chosen.

A rule F satisfies Unanimity condition if,

$$[\forall i \in N \quad X \cap D_i(x) = \emptyset] \Rightarrow x \in C(X),$$

or, equivalently,

$$[n^+(x, \overrightarrow{P}) = n] \Rightarrow x \in C(X).$$

Now the following theorem can be introduced.

Theorem 2 *(Aleskerov, 1992)* $\Lambda^{ND^+} \subset \Lambda^U \subset \Lambda^{NI^+}$ *and* $\Lambda^{ND^-} \subset \Lambda^{NI^-}$.

11) *No veto power1* $(\mathbf{NVP_1})$

No veto power1 condition requires that, given a profile if there exists an alternative that is preferred by every voter but one to a given alternative than the given alternative must be out of social decision chosen.

A rule F satisfies No veto power1 condition if,

$$[\forall i \in N \backslash \{j\} \quad xP_iy \text{ and } yP_jx] \Rightarrow \; y \notin C(X).$$

12) *No veto power2* $(\mathbf{NVP_2})$

No veto power2 condition requires that, given a profile if there exists an alternative x that is preferred by every voter but one to a given alternative y than the y may be in social decision but can not drive others (especially x) out of social decision for otherwise this one individual will have the right to veto x.

A rule F satisfies No veto power2condition if,

$$[\forall i \in N\backslash\{j\} \quad xP_iy \text{ and } yP_jx] \Rightarrow \ \{y\} \neq C(X).$$

Theorem 3 $\Lambda^{NVP_1} \subseteq \Lambda^{NVP_2}$.

Proof : Assume that $F \in \Lambda^{NVP_1}$. Then $\forall \overrightarrow{P} \in \mathcal{W} \quad \forall X \in \mathcal{A}$ if $\forall i \in N\backslash\{j\}$ then xP_iy and yP_jx, then $\{y\} \neq C(X)$ hence $F \in \Lambda^{NVP_2}$.

13) *Axiom of Reinforcement* (**RA**)

Axiom of Reinforcement deals with two joinig subsocieties. If from both of the subsocieties there are common alternatives in (sub-)social decisions then from the society as a whole all those and only those alternatives should be the social decision.

A rule F satisifes Reinforcement Axiom if $\forall \omega_1, \omega_2 \subseteq N$ such that $N = \omega_1 \cup \omega_2$, $\omega_1 \cap \omega_2 = \emptyset$,

$$C_{\omega_1}(A) \cap C_{\omega_2}(A) \neq \emptyset \Rightarrow C_N(A) = C_{\omega_1}(A) \cap C_{\omega_2}(A).$$

14) *Axiom of Participation* (**PA**)

Axiom of Participation deals with an additional individual joining to the existing society. If someone else joins the society, then either he can not get a previously chosen alternative out of the social decision or he can but this time replacing it with an alternative which he/she more likes. Notice that 'or' is meant to be inclusive in the definition above.

A rule F satisifes Participation Axiom if,

$$x \in C_N(A) \Rightarrow \ [(x \in C_{N\cup\{j\}}(A)) \text{ or } (y \in C_{N\cup\{j\}}(A) \text{ and } yP_jx)].$$

4.1.2 Normative Conditions for Functional Voting Rules

In what follows, definitions of NCs for FVRs are introduced. From now on the abbreviations given in parentheses are going to be used for the names of the conditions and for the FVRs that satisfy the condition Λ^Q will be used such as Λ^M for classes of rules that satisfy Monotonicity. Since the social decision is a choice function, only $C(X)$ will be used instead of $F(X, \overrightarrow{C})$. For definitions of NCs for FVRs every statement is true $\forall \overrightarrow{C}, \overrightarrow{C'} \in \mathcal{C}^n$ and $\forall x \in A, \quad \forall X \in \mathcal{A}$, such that $x \in X$, as necessary. (When an addition is necessary this will be done explicitly).

1) *Locality* (**L**)

A rule F satisifes locality if,

$$[V(x,X;\overrightarrow{C}) = V(x,X;\overrightarrow{C'})] \Rightarrow \ [\ x \in C(X) \Leftrightarrow x \in C'(X)].$$

2) *Positive non-imposedness* ($\mathbf{NI^+}$)

A rule F satisifes Positive non-imposedness if,

$$\exists \overrightarrow{C} \in \mathcal{C}^n \text{ such that } x \in C(X).$$

3) *Negative non-imposedness* ($\mathbf{NI^-}$)

A rule F satisifes Negative non-imposedness if,

$$\exists \{C_i(\cdot)\} \in \mathcal{C}^n \text{ such that } x \notin C(X).$$

Remark 5: All the Functional Voting Rules that I will consider satisfy $\mathbf{NI^+}$ and $\mathbf{NI^-}$ which can be easily checked by the reader.

3) *Monotonicity (*$\mathbf{M}$)

A rule F satisifes monotonicity if,

$$[V(x,X;\{C_i(\cdot)\}) \subseteq V(x,X;\{C_i'(\cdot)\})] \Rightarrow \ [x \in C(X) \Rightarrow x \in C'(X)].$$

4) *Neutrality1* ($\mathbf{Ne_1}$)

Let $\sigma : A \rightarrow A$ be a bijection and $\sigma(X) = \{\sigma(x) \in A : x \in X\}$. A rule F satisifes Neutrality1 if,

$$[\forall i \in N \quad C_i'(\sigma(X)) = \sigma(C_i(X))] \Rightarrow [C'(\sigma(X)) = \sigma(C(X))].$$

Remark 6: All the Functional Voting Rules that I will consider satisfy $\mathbf{Ne_1}$ which can be easily checked by the reader.

5) *Neutrality* ($\mathbf{Ne_2}$)

A rule F satisifes neutrality if $\forall \overrightarrow{C}, \overrightarrow{C'} \in \mathcal{C}^n$ and $\forall x', x'' \in A, \quad \forall X', X'' \in \mathcal{A}$, such that $x' \in X'$ and $x'' \in X''$,

$$[V(x',X';\{C_i'(X')\}) = V(x'',X'';\{C_i''(X'')\})] \Rightarrow$$
$$[x' \in C'(X') \Leftrightarrow x'' \in C''(X'')].$$

Now the following theorem is introduced.

Theorem 4 *(Aleskerov, forthcoming)* $\Lambda^{\mathbf{M}} \subset \mathbf{\Lambda^L}$ *and* $\Lambda^{\mathbf{Ne_2}} \subset \mathbf{\Lambda^L}$.

6) *Anonymity* (**An**)

A rule F satisifes anonymity if $\forall \{C_i(\cdot)\} = \overrightarrow{C}, \{C_{\eta(i)}(\cdot)\} \in \mathcal{C}^n$ and $\forall X \in \mathcal{A}$ where $\eta : N \to N$ is a bijection,

$$F(X, \overrightarrow{C}) = C(X) = C'(X) = F(X, \{C_{\eta(i)}(\cdot)\}).$$

Remark 7: All the Functional Voting Rules that I will consider but Voting with Veto procedure satisfies **An** which can be easily checked by the reader. The case of Voting with Veto is investigated in the next Chapter.

Definition 4 *The class of rules that satisifes* $\mathbf{NI^+}, \mathbf{NI^-}, \mathbf{M}$ *and* $\mathbf{Ne_2}$ *are called the Central Class denoted by* $\Lambda^{\mathbf{C}}$*, i.e.,* $\Lambda^{\mathbf{NI^+}} \cap \Lambda^{\mathbf{NI^-}} \cap \Lambda^{\mathbf{M}} \cap \Lambda^{\mathbf{Ne_2}} = \Lambda^{\mathbf{C}}$. *The rules from Central Class that satisfy* **An** *are called the Symmetrically Central Class denoted by* $\Lambda^{\mathbf{SC}}$*, i.e.,* $\Lambda^{\mathbf{C}} \cap \Lambda^{\mathbf{An}} = \Lambda^{\mathbf{SC}}$.

7) *Positive unanimity (Pareto principle)* ($\mathbf{U^+}$)

A rule F satisifes Positive unanimity if,

$$\forall i \in N \quad x \in C_i(X) \Rightarrow x \in C(X).$$

8) *Negative unanimity* ($\mathbf{U^-}$)

A rule F satisifes Negative unanimity if,

$$\forall i \in N \quad x \notin C_i(X) \Rightarrow x \notin C(X).$$

Theorem 5 *(Aleskerov, forthcoming) The class of rules that satisfy Positive (resp. Negative) Unanimity is strictly embedded in the class of rules that satisfy Positive (resp. Negative) non-imposedness, i.e.,* $\Lambda^{U^+} \subset \Lambda^{NI^+}$ *and* $\Lambda^{U^-} \subset \Lambda^{NI^-}$.

9) *No veto power* (NVP)

A rule F satisifes No veto power if for any profile of single valued choice functions, $\forall \{C_i(\cdot)\} \in \widehat{\mathcal{C}}^n$, and $\forall x \in A, \quad \forall X \in \mathcal{A}$, such that $x \in X$,

$$[V(x, X; \{C_i(\cdot)\}) = n - 1 \text{ and } V(y, X; \{C_i(\cdot)\}) = 1] \Rightarrow y \notin C(X).$$

10) *Axiom of Reinforcement*

A rule F satisifes Axiom of Reinforcement $\forall \omega_1, \omega_2 \subseteq N$ such that $N = \omega_1 \cup \omega_2$, $\omega_1 \cap \omega_2 = \emptyset$ if,

$$C_{\omega_1}(A) \cap C_{\omega_2}(A) \neq \emptyset \Rightarrow C_N(A) = C_{\omega_1}(A) \cap C_{\omega_2}(A).$$

4.1.3 Normative Conditions for Social Decision Rules

In what follows, definitions of NCs for SDRs are introduced. These definitions will be given in terms of the set of all alternatives A but they equally apply for any presentation $X \in \mathcal{A}$. From now on the abbreviations given in parentheses are going to be used for the names of the conditions and for the SDRs that satisfy the condition Λ^Q will be used such as Λ^M for classes of rules that satisfy Monotonicity. Since the social decision is a binary relation, only P will be used instead of $F(\overrightarrow{P})$. For definitions of NCs for SDRs every statement is true $\forall \overrightarrow{P}, \overrightarrow{P'} \in \mathcal{W}^n$ and $\forall x, y, z, w \in A$ as necessary. (When an addition is necessary this will be done explicitly).

1) *Quasilocality (Aleskerov and Vladimirov, 1986)*

Quasilocality condition is different from Locality condition introduced by Aizeman and Aleskerov (1984) in the sense that it not only considers the set of voters that include pair (x, y) in his preference but also considers the ones that include (y, x). It requires if set of voters for (x, y) and set of voters for (y, x) are both kept the same between two profiles $\overrightarrow{P}, \overrightarrow{P'}$ then the pair (x, y) is included in P if and only if it is included in P'.

A rule F satisifes locality if,

$$[V(x, y; \overrightarrow{P}) = V(x, y; \overrightarrow{P'}) \text{ and } V(y, x; \overrightarrow{P}) = V(y, x; \overrightarrow{P'})]$$

$$\Rightarrow [(x, y) \in P \Leftrightarrow (x, y) \in P'].$$

2) *Positive non-imposedness* ($\mathbf{NI}^+$)

A rule F satisifes Positive non-imposedness if,

$$\exists \overrightarrow{P} \in \mathcal{W}^n \text{ such that } (x, y) \in P.$$

3) *Negative non-imposedness* ($\mathbf{NI}^-$)

A rule F satisifes Negative non-imposedness if,

$$\exists \overrightarrow{P} \in \mathcal{W}^n \text{ such that } (x, y) \notin P.$$

Remark 8: All the Social Decision Rules that I will consider satisfy $\mathbf{NI}^+$ and $\mathbf{NI}^-$ which can be easily checked by the reader.

4) *Monotonicity (**M**)*

Monotonicity condition has been adapted to the condition of Quasilocality rather than Locality.

A rule F satisifes Monotonicity if,

$$(x,y) \in P \text{ and } [V(x,y;\overrightarrow{P}) \subseteq V(x,y;\overrightarrow{P'}) \text{ and } V(y,x;\overrightarrow{P'}) \subseteq V(y,x;\overrightarrow{P})]$$

$$\Rightarrow [(x,y) \in P'].$$

5) *Neutrality1 (***Ne₁***)*

Let $\sigma : A \to A$ be a bijection and let $\sigma(X) = \{\sigma(x) \in A : x \in X\}$. A rule F satisifes Neutrality1 if,

$$[\forall i \in N \quad \sigma(L_i(x)) = L'_i(\sigma(x)) \text{ and } \sigma(D_i(x)) = D'_i(\sigma(x))]$$

$$\Rightarrow [xPy \Leftrightarrow \sigma(x)P'\sigma(y) \text{ where } P' = F(A, \sigma(\overrightarrow{P}))].$$

Remark 9: All the Social Decision Rules that I will consider satisfy $\mathbf{Ne_1}$ which can be easily checked by the reader.

6) *Neutrality2* ($\mathbf{Ne_2}$)

Neutrality2 condition has been adapted to the condition of Quasilocality rather than Locality.

A rule F satisifes Neutrality2 if[10],

$$[\forall i \in N \quad V(x,y;\overrightarrow{P}) = V(z,w;\overrightarrow{P'}) \text{ and } V(y,x;\overrightarrow{P}) = V(w,z;\overrightarrow{P'})]$$

$$\Rightarrow [(x,y) \in P \Leftrightarrow (z,w) \in P].$$

Now the following theorem can be introduced.

Theorem 6 *(Aleskerov and Vladimirov, 1986)* $\mathbf{\Lambda^M} \subset \mathbf{\Lambda^L}$ *and* $\mathbf{\Lambda^{Ne_2}} \subset \mathbf{\Lambda^L}$.

7) *Anonymity* (**An**)

A rule F satisifes Anonymity if $\forall \overrightarrow{P}, \overrightarrow{P'} \in \mathcal{W}^n$ and $\forall X \in \mathcal{A}$ where $\eta : N \to N$ is a bijection and $\{P_{\eta(i)}\}_{\eta(i) \in N} = \overrightarrow{P'}$, one always has $P = P'$.

Remark 10: All the Social Decision Rules that I will consider satisfy **An** which can be easily checked by the reader.

8) *Positive Pareto* ($\mathbf{PP^+}$)

Positive Pareto condition requires that if no voter includes a pair (y,x) and at least one includes (x,y) in his preference then in the social decision the rule should include (x,y).

[10]Since one can change the order of x and y, and z and w in the assumption of the condition , it is obvious that the conclusion of the condition Ne_2 can be written as $(y,x) \in P \Leftrightarrow (w,z) \in P$.

A rule F satisifes Positive Pareto if,

$$[\forall i \in N \quad (y,x) \notin P_i \text{ and } \exists i_0 \in N \text{ such that } (x,y) \in P_{i_0}] \\ \Rightarrow \ [(x,y) \in P].$$

9) *Negative Pareto* $(\mathbf{PP^-})$

Negative Pareto condition requires that if no voter includes a pair (x,y) in his preference then in the social decision the rule should not include (x,y).

A rule F satisifes Negative Pareto if,

$$[\forall i \in N \quad (x,y) \notin P_i] \Rightarrow \ [(x,y) \notin P].$$

Remark 11: All the Social Decision Rules that I will consider satisfy $\mathbf{PP^-}$ which can be easily checked by the reader.

Definition 5 *The class of social decision rules that satisfy* $\mathbf{NI^+}, \mathbf{NI^-}, \mathbf{M}$, $\mathbf{Ne_2}$ *and* $\mathbf{PP^-}$ *are called the Central Class denoted by* $\Lambda^{\mathbf{C}}$*, i.e.,* $\Lambda^{\mathbf{NI^+}} \cap \Lambda^{\mathbf{NI^-}} \cap \Lambda^{\mathbf{M}} \cap \Lambda^{\mathbf{Ne_2}} \cap \Lambda^{\mathbf{PP^-}} = \Lambda^{\mathbf{C}}$*. The rules from Central Class that satisfy* $\mathbf{An}$ *are called the Symmetrically Central Class denoted by* $\Lambda^{\mathbf{SC}}$*, i.e.,* $\Lambda^{\mathbf{C}} \cap \Lambda^{\mathbf{An}} = \Lambda^{\mathbf{SC}}$.

10) *Positive unanimity* $(\mathbf{U^+})$

Positive unanimity condition requires that if all the voters include a pair (x,y) in their preferences then in the social decision the rule should include (x,y).

A rule F satisifes Positive unanimity if,

$$[\forall i \in N \quad (x,y) \in P_i] \Rightarrow \ [(x,y) \in P].$$

11) *Negative unanimity* $(\mathbf{U^-})$

Negative unanimity condition requires that if all the voters include a pair (y,x) in their preferences then in the social decision the rule should not include (x,y).

A rule F satisifes Negative unanimity if,

$$[\forall i \in N \quad (y,x) \in P_i] \Rightarrow \ [(x,y) \notin P].$$

12) *No veto Power* $(\mathbf{NVP})$

No veto power condition requires that if all but one of the voters include a pair (x,y) in their preferences then in the social decision the rule should include (x,y).

A rule F satisifes No veto power if,

$$[\forall j \in N \quad \forall i \in N \backslash \{j\} \quad (x,y) \in P_i] \Rightarrow \ [(x,y) \in P].$$

4.2 Rationality constraints

Normative conditions define subsets in the space of all SCPs. This section will introduce Rationality Constraints (RCs) for Social Choice Procedures (SCPs) and some related theorems. RCs are defined on choice functions or binary relations, that is the domains and the ranges of the SCPs. To check whether a SCP 'satisfies' a given RC, the domain of the SCP will be fixed in terms of satisfying some RCs, then the social decision will be checked to satisfy the given RC. When checking a FVR or a SDR the domain will necessarily satisfy the given RC since the social decision has the same structure with individual opinions.

Two of the choice models introduced have social decision in the form of a choice function (Social Choice Correspondences and Functional Voting Rules) and one in the form of a binary relation (Social Decision Rules). Depending on the form of the social decision and the individual opinions in the choice model, corresponding RCs will be introduced in the order above as in the previous section for the case of Normative Conditions. While introducing definitions some basic theorems will be provided.

In the next subsection RCs for choice functions are defined and after this some theorems about their mutual relations in the space of all choice functions $\mathcal{C}$ are given. Then these theorems are utilized to establish relationships between SCCs and then FVRs satisfying these RCs. Finally, the method for checking a SCC and a FVR to satisfy a given rationality constraint is given. In the third subsection, the RCs for binary relations are given and after this again some theorems about their mutual relationships are given. Then these theorems are utilized to establish relationships between SDRs satisfying these RCs. Finally, the method for checking a SDR to satisfy a given rationality constraint is given.

4.2.1 Rationality Constraints for Social Choice Correspondences and Functional Voting Rules

Definitions of Rationality Constraints In what follows, definitions of rationality constraints for choice functions are introduced. From now on the abbreviations given in parentheses are going to be used for the names of the constraints.

1) *Non-emptiness*[11] (**NE**)

If the function $C(\cdot)$ never gives empty choice as a social decision, then it satisfies condition **NE**, i.e., $C(\cdot) \in \mathbf{NE} \Leftrightarrow$

$$\forall X \in \mathcal{A} \quad C(X) \neq \emptyset.$$

[11] The case of non-emptiness is investigated by many authors. The reader can be referred to (Arrow, 1959), (Sen, 1970), (Sen, 1974), (Samuelson, 1938),(Chernoff, 1954).

2) *Weak non-resoluteness* (**WNR**) (Popov, B.V. and L.N. El'kin, 1989)

If the function $C(\cdot)$ chooses a proper subset from presentation A, then it satisfies condition **WNR**, i.e., $C(\cdot) \in \textbf{WNR} \Leftrightarrow$

$$C(A) \subset A.$$

Conditions 3 to 8 are from (M. Aizerman and F. Aleskerov, 1995) and from references cited there.

3) *Heredity* (**H**)

A choice function $C(\cdot)$ satisfies **H**, if an alternative that is chosen from a given set is always chosen from any subset of this set in which it exists. This is because the alternative is facing no different alternatives in the latter case and it once 'succeeded' among them so for sake of 'consistency' it should succeed in the 'smaller' set. Formally, $C(\cdot) \in \textbf{H} \Leftrightarrow$

$$\forall X', X \in \mathcal{A}, \quad X' \subseteq X \Rightarrow C(X') \supseteq C(X) \cap X'.$$

This condition, for example, implies that if a player is chosen among the 'best' players of the country then he/she should be chosen among the best from his/her city. Note that the condition **H** does not rule out the possibility that the choice from the smaller set X' also includes the alternatives not chosen from the larger set X, and that for $X' \cap C(X) = \emptyset$ the condition **H** imposes no constraints on $C(X')$.

4) *Arrow's Choice Axiom* (**ACA**)

If a choice function $C(\cdot)$ satisfies **ACA** then alternatives chosen from a given set and only those alternatives (to the extent that they exist) must always be chosen from any subset of this set. Furthermore, if choice from a 'larger' set is empty then choice from any of its subsets will be empty. Formally, $C(\cdot) \in \textbf{ACA} \Leftrightarrow$

$$\forall X', X \in \mathcal{A}, \quad X' \subseteq X \Rightarrow \left\{ \begin{array}{l} \text{if } C(X) = \emptyset, \text{ then } C(X') = \emptyset, \\ \text{if } C(X) \cap X' \neq \emptyset, \text{ then } C(X') = C(X) \cap X'. \end{array} \right\}$$

This condition, for example, implies that if a group of students are chosen among the 'best' students of the school then each should be chosen as the best from his/her class and no one else which is not best in school should be chosen as best from their classes. Note that this condition does not constrain choice from 'smaller' set X', if choice from the 'big' set X does not contain any of its alternatives but non-empty, that is if $C(X) \neq \emptyset$, but $C(X) \cap X' = \emptyset$.

5) *Concordance* (**C**)

A choice function $C(\cdot)$ satisfies **C**, if an alternative that is chosen from two given presentations is always chosen from the union of these two sets, i.e., $C(\cdot) \in \mathbf{C} \Leftrightarrow$

$$\forall X', X \in \mathcal{A}, \quad X = X' \cup X'' \Rightarrow C(X) \supseteq C(X') \cap C(X'').$$

This condition, for example, implies that if a player is chosen among the 'best' players of both of given two classes, then he/she should be chosen the best from the unification of two classes. Note that the choice from $X = X' \cup X''$ may have alternatives that are not chosen at separate presentations of X' and X'', even if they are present both in X' and X''.

6) *Independence of Outcast* (**O**)

A choice function $C(\cdot)$ satisfies **O**, if a subset X' of a given a presentation X contains the choice set $C(X)$ of X, then exactly the same choice set is always chosen from X'. Equivalently, if a subset X' of X is eliminated from X, where X' does not contain any of the alternatives from $C(X)$, then choice from the remaining set equals $C(X)$ then $C(\cdot) \in \mathbf{O}$.

$$\forall X, X' \in \mathcal{A} \quad C(X) \subseteq X' \subseteq X \Rightarrow C(X') = C(X).$$

or, equivalently,

$$\forall X, X' \in \mathcal{A} \quad X' \subseteq X \setminus C(X) \Rightarrow C(X \setminus X') = C(X).$$

This condition, for example, implies that if a group of students are chosen as the 'best' students of the bigger class and if they are again all together in a smaller class, again exactly they must be chosen as the best of the smaller class.

7) *Inverse Condorcet Principle* (**Con**$^-$)

A choice function $C(\cdot)$ satisfies **Con**$^-$, if an alternative $x \in A$, chosen from a presentation X, is chosen from every pairwise presentation contained in X, i.e., $C(\cdot) \in$ **Con**$^- \Leftrightarrow$

$$\forall X \in \mathcal{A}, \quad \forall y \in X \quad x \in C(X) \Rightarrow x \in C(\{x, y\}),$$

or, equivalently,

$$C(X) \subseteq \bigcap_{y \in x} C(\{x, y\}).$$

For example, the finalist teams of the basketball league must not have been beaten by any other team in the league.

8) *Direct Condorcet Principle* (**Con**$^+$)

A choice function $C(\cdot)$ satisfies **Con**$^+$, if an alternative $x \in A$, chosen from every pairwise presentation contained in X is always chosen from a presentation X, i.e., $C(\cdot) \in \mathbf{Con}^+ \Leftrightarrow$

$$\forall X \in \mathcal{A}, \quad \forall y \in X \quad x \in C(\{x, y\}) \Rightarrow x \in C(X),$$

or, equivalently,

$$\bigcap_{y \in x} C(\{x, y\}) \subseteq C(X).$$

For example, a team in the football league that have beaten every other team in the league must be the champion of the league.

9) *Dual Heredity* (**H**$^-$) (F. Aleskerov, J. Duggan 1993)

A choice function $C(\cdot)$ satisfies Dual Heredity constraint **H**$^-$, if from a subset of a presentation X a subset of the choice set $C(X)$ is chosen, i.e.,

$$\forall X', X \in \mathcal{A} \quad X' \subseteq X \Rightarrow C(X') \subseteq C(X).$$

As follows immediately from the definitions, Dual Heredity condition and Heredity Condition are dual, that is, if the function $C(\cdot)$ satisfies **H**, the function $\overline{C}(X) = X \setminus C(X)$ satisfies **H**$^-$, and vice versa.

10) **ACA** *in strong version (***ACA***)(M. Aizerman and Petnisk, 1995)

A choice function $C(\cdot)$ satisfies **ACA***, if there exists a set A^* in A such that choice set from any presentation contains those and only those elements that are also in A^*, i.e.,

$$\forall X \in \mathcal{A} \quad C(X) = X \cap A^* \text{ where } A^* \subseteq A.$$

11) *Path Independence (***PI**) (Plott, 1973)

A choice function $C(\cdot)$ satisfies **PI**, if it is independent of the order that presentations are presented or whether they are presented one by one or all together, i.e.,

$$\forall X_1, X_2 \in \mathcal{A}, \quad C(X_1 \cup X_2) = C(C(X_1) \cup X_2),$$

or, equivalently,

$$C(X_1 \cup X_2) = C(C(X_1) \cup C(X_2)).$$

12) *Fixed Point* (**FP**) (Aleskerov, 1992)

A choice function $C(\cdot)$ satisfies **FP**, if it always chooses an alternative x from a given presentation X and from all subsets of X as long as x exists in these subsets, i.e.,

$$\forall X', X \in \mathcal{A}, \quad \exists x \in X \text{ such that } [x \in X' \subseteq X \Rightarrow x \in C(X')].$$

or, equivalently,

$$\forall X', X \in \mathcal{A} \text{ such that } x \in X' \subseteq X \quad x \in \bigcap_{X'} C(X') \neq \emptyset.$$

13) *Weak Axiom of Revealed Preferences (WARP) (Samuelson, 1938) (Houthakker, 1950)*

To define **WARP** it is necessary to construct a binary relation **G** in the following manner.

$$x\mathbf{G}y \Leftrightarrow \exists X \in \mathcal{A} \text{ such that } [x \in C(X) \quad \text{and} \quad y \in X \backslash C(X)].$$

A choice function $C(\cdot)$ satisfies **WARP** if **G** is acyclic[12].

For example the choice function of a housewife violates **WARP** if she chooses to buy only an apple from the first set of fruits {cherry,apple} but chooses to buy banana and cherry from the next set of fruits {banana,apple,cherry, grapes}.

4.2.2 Some theorems about rationality constraints(for SCCs and FVRs) and their mutual relationships

Each of the rationality constraints separate the class of all choice functions $\mathcal{C}$ into two, that is the ones satisfying it and others. These classes are interrelated in many ways. The following theorems reveal these relationships. Note that the letter(s) denoting the rationality constraint will also be used to denote the set of choice functions which satisfy it and a bar above the letter will denote the set of choice functions which do not satisfy the rationality constraint. For example **H** will denote the set of choice functions which satisfy **H**, and $\overline{\mathbf{H}}$ will denote the set of choice functions which do not satisfy **H**.

There are eight possible domains related to rationality constraints **H**, **C** and **O**. The following theorem shows how they are related.

Theorem 7 *(M. Aizerman and F. Aleskerov, 1995) (i) In the space $\mathcal{C}$ the properties* **H**, **C** *and* **O** *are independent, that is, all eight domains*

$$\mathbf{H} \cap \mathbf{C} \cap \mathbf{O};\ \overline{\mathbf{H}} \cap \mathbf{C} \cap \mathbf{O}, \ldots, \overline{\mathbf{H}} \cap \overline{\mathbf{C}} \cap \overline{\mathbf{O}}$$

[12] Acyclicity of a binary relation is that there exists no 'cycles' of any length m throughout the binary relation, i.e., G is acyclic iff $\quad \forall x_1, ..., x_m \in A \quad \rceil \exists x_1 G x_2 G ... G x_m G x_1$.

are not empty.

(*ii*) *The property* **ACA** *is stronger than each of the properties* **H**, **C** *and* **O**, *that is, the domain* **ACA** *lies strictly within the intersection* $\mathbf{H} \cap \mathbf{C} \cap \mathbf{O}$.

(*iii*) *In the subspace* $\mathcal{C}^+$ *of non-empty choice functions the domains* **H**, **C**, **O**, *and* **ACA** *are related in the same way.*

(*iv*) *In the subspace* $\widehat{\mathcal{C}}$ *of single valued choice functions the domains* **H**, **ACA**, *and* **O** *coincide, thus, making up the domain* $\mathbf{H} = \mathbf{O} = \mathbf{ACA}$ *located strictly within the domain* **C**, *i.e.,* $\mathbf{H} - \mathbf{O} - \mathbf{ACA} \subset \mathbf{C}$.

The conjunction of two previously defined RCs, $\mathbf{Con}^-$ and $\mathbf{Con}^+$ is called the Condorcet Principle and denoted by PC. Next two theorems are about the PC and conditions **H** and **C**.

Theorem 8 *(Aizerman and Aleskerov, 1995)* $\mathbf{H} \subseteq \mathbf{Con}^-$ *and* $\mathbf{C} \subseteq \mathbf{Con}^+$.

Theorem 9 *(Aizerman and Aleskerov, 1995) The domain of functions satisfying the Condorcet principle coincides with the domain distinguished by the conjunction of the conditions* **H** *and* **C**, *i.e.,* $\mathbf{PC} = \mathbf{Con}^- \cap \mathbf{Con}^+ = \mathbf{H} \cap \mathbf{C}$.

Condition **ACA** in strong version ($\mathbf{ACA}^*$) is related to other conditions introduced above by the following theorem.

Theorem 10 *(Aizerman and Aleskerov, 1995)The class of choice functions satisfying* $\mathbf{ACA}^*$ *is strictly embedded into the class* **ACA**, *and the functions of this class are defined by the conjunction of the dual Heredity and Heredity axioms, i.e.,* $\mathbf{ACA}^* = \mathbf{H} \cap \mathbf{H}^- \subset \mathbf{ACA}$.

Remark 12: Note that in the case of nonempty choice the dual Heredity condition defines a single 'point' in the space of choice functions, the function $C(X) = X$ for any $X \in \mathcal{A}$. In fact, since in this case $C(\{x\}) = \{x\}$ for any x and in virtue of Condition $\mathbf{H}^-$, $x \in C(X)$ for all x and X such that $x \in X \in \mathcal{A}$, that is, $C(X) = X$. Similarly, Condition $\mathbf{ACA}^*$ defines the same function, because the set A^* coincides with the set A for nonempty choice.

Now the relation between the condition **PI** and the domains **H**, **C** and **O** in $\mathcal{C}$ is in order.

Theorem 11 *(Aizerman and Aleskerov, 1995) The domain of functions satisfying the path-independence condition* (**PI**) *coincides in* $\mathcal{C}$ *with the intersection of* **H** *and* **O**.

Theorem 12 *(Aleskerov, 1992) The class of choice functions that satisfy condition* **FP** *is strictly embedded in the class of choice functions that satisfy condition* **NE**, *i.e.,* $\mathbf{FP} \subset \mathbf{NE}$. *Moreover, the set of all choice functions that satisfy condition* **H** *and condition* **NE** *is strictly embedded in the class of choice functions that satisfy condition* **FP**, *i.e.,* $\mathbf{H} \cap \mathbf{NE} \subset \mathbf{FP}$.

Theorem 13 *(Aizerman and Aleskerov, 1995) The class of choice functions that satisfy ACA can be rationalized by a weak order P where* $\forall X \in \mathcal{A} \quad C(X) = \{x \in X : \rceil \exists y \in X$ *such that* $yPx\}$.

Theorem 14 *(Aizerman and Aleskerov, 1995) The domain of functions satisfying Arrow 's Choice Axiom* (**ACA**) *is strictly embedded in the class of choice functions which satisfy Weak axiom of revealed preferences* (**WARP**)*, i.e.,* **ACA** $\subset$ **WARP**.

Checking for a SCC and a FVR for satisfying a Rationality Constraint In each of the definitions, the social decision is denoted by $C(\cdot)$ which can be defined as $C(X) = F(X, \overrightarrow{P})$ for SCCs or $C(X) = F(X, \overrightarrow{C})$ for FVRs.

For a SCC to be checked for satisfying a rationality constraint the domain of definition $Q_{\mathbf{d}}$ is assumed that the individual opinions are in the form of weak orders, i.e., $\forall i \in N \quad P_i \in \mathcal{W}$, or in short $\overrightarrow{P} \in \mathcal{W}^n$ where $\mathcal{W}$ is the set of all weak orders. Note that, a weak ordered preference enables the individual to establish either the 'equivalency' of any given two alternatives or 'dominance' of one on another.

For FVRs, individual opinions are in general assumed to be satisfying Arrow 's Choice Axiom, Non-emptiness and $Q_{\mathbf{r}}$, i.e., $Q_{\mathbf{d}} = ACA \cap NE \cap Q_{\mathbf{r}} = ACA^{+} \cap Q_{\mathbf{r}}$. The two exceptions to this will be made when checking for Arrow 's Choice Axiom in strong version (ACA^{*}) and dual Heritage constraints. For sake of simplicity, the domain of definition $Q_{\mathbf{d}}$ which will be used for checking for a rationality constraint will be added to the definition of the raionality constraint.

Definition 6 *A given SCP is said to satisfy a given rationality constraint, if and only if for any given profile from the domain of definition as stated above, the social decision constructed according to the given SCP (which is a choice function or a binary relation) satisfies the given rationality constraint. The set of all SCPs satisfying a given rationality constraint Q will be denoted by* $\Lambda(Q_d, Q)$.

For example for SCCs that satisfy the constraint H under the domain restricted to RC Q, $\Lambda(Q, H)$ will be used. When $Q_d = \mathcal{W}^n$, this will be considered as default and hence $\Lambda(\mathcal{W}^n, H) = \Lambda(H)$ will be used.

Similarly, for FVRs that satisfy the constraint H under the domain restricted to RC Q, $\Lambda(Q, H)$ will be used. When $Q_d = (\mathbf{ACA}^{+})^n$, this will be considered as default and hence $\Lambda(ACA^{+}, H) = \Lambda(H)$ will be used.

Consider SCCs and the following example which explains how a SCC can belong to the class of rules that satisfy a given rationality constraint.

Example 15 *Let the rationality constraint under concern be non-emptiness* (NE) *of social decision. Given a social choice correspondence* F*, and a profile of weak orders* $\overrightarrow{P}$*, if this SCC always leads to a non-empty choice set given any presentation, i.e.,* $\forall X \in \mathcal{A} \quad C(X) \neq \emptyset$*, then this SCC satisfies the condition of non-emptiness, i.e.,* $F \in \Lambda(NE)$.

Now consider FVRs and the following example which explains how a FVR can belong to the class of rules that satisfy a given rationality constraint.

Example 16 *Let the rationality constraint under concern be weak non-resoluteness* (WNR) *of social decision. Given a FVR* F*, and a profile of choice functions which satisfy Arrow 's Choice Axiom, Non-emptiness* (ACA^+) *and* WNR*, if this FVR always leads to a proper subset of presentation* A*, i.e.,* $\forall i \in N \quad C_i(\cdot) \in ACA^+ \cap WNR \quad C(A) \subset A$*, then this FVR satisfies the condition of weak non-resoluteness, i.e.,* $F \in \Lambda(ACA^+ \cap WNR, WNR)$.

Theorem 15 *In the space of social choice correpondences* Λ_{SCC}*, the classes satisfying all the rationality constraints cited above are related in the same manner as corresponding rationality constraints are related hence the following are true:*
(i) $\Lambda(ACA) \subset \Lambda(H) \cap \Lambda(C) \cap \Lambda(O) \cap \Lambda(Con^-) \cap \Lambda(Con^+) \cap \Lambda(PI) \cap \Lambda(WARP)$,
(ii) $\Lambda(H) \subset \Lambda(Con^-)$ *and* $\Lambda(C) \subset \Lambda(Con^+)$ *where* $\Lambda(H) \cap \Lambda(C) = \Lambda(Con^-) \cap \Lambda(Con^+)$,
(iii) $\Lambda(H) \cap \Lambda(O) = \Lambda(PI)$,
(iv) $\Lambda(H) \cap \Lambda(NE) \subset \Lambda(FP) \subset \Lambda(NE)$,
(v) $\Lambda(H) \cap \Lambda(H^-) = \Lambda(ACA^*)$.

Proof : For each of the above rationality constraints (WNR is not considered here) the domain of definition is the same, that is $Q_d = \mathcal{W}^n$. Then the relationships among the rules concerning these rationality constraints are reduced to relationships in the range of the rule which is the set of choice functions, which can be examined by the same theorems. □

Theorem 16 *In the space of functional voting rules* Λ_{FVR}*, the classes satisfying* $\Lambda(ACA)$, $\Lambda(H)$, $\Lambda(C)$, $\Lambda(O)$, $\Lambda(Con^-)$, $\Lambda(Con^+)$, $\Lambda(PI)$, $\Lambda(WARP)$, $\Lambda(NE)$ *and* $\Lambda(FP)$ *are related in the manner as corresponding rationality constraints are related hence the following are true.*
(i) $\Lambda(ACA) \subset \Lambda(H) \cap \Lambda(C) \cap \Lambda(O) \cap \Lambda(Con^-) \cap \Lambda(Con^+) \cap \Lambda(PI) \cap \Lambda(WARP)$,
(ii) $\Lambda(H) \subset \Lambda(Con^-)$ *and* $\Lambda(C) \subset \Lambda(Con^+)$ *where* $\Lambda(H) \cap \Lambda(C) = \Lambda(Con^-) \cap \Lambda(Con^+)$,
(iii) $\Lambda(H) \cap \Lambda(O) = \Lambda(PI)$, *(iv)* $\Lambda(H) \cap \Lambda(NE) \subset \Lambda(FP) \subset \Lambda(NE)$.

Proof : For each of the rationality constraints ACA, $H, C, O, Con^-, Con^+, PI$, and $WARP$, the domain of definitions are the same, that is $Q_d = (ACA^+)^n$,since they are all supersets of ACA^+ by theorems **??**, **??**, **??** and **??**. Then the relationships among the rules concerning these rationality constraints are reduced to relationships in the range of the rule which is the set of choice functions, which can be examined by the same theorems. □

Theorem 17 *In the space of functional voting rules* Λ_{FVR}*, the classes satisfying* $\Lambda(ACA^*, ACA^*)$, $\Lambda(ACA^*, H)$ *and* $\Lambda(ACA^*, H^-)$ *are related in the following manner:* $\Lambda(ACA^*, H) \cap \Lambda(ACA^*, H^-) = \Lambda(ACA^*, ACA^*)$.

Proof : Since $ACA \cap H^- = ACA^*$ by Theorem **??**, when checking for H^- and ACA^* the domain of definition will be ACA^*. Since domain of definition is the same for H^- and ACA^*, the relation between rules satisfying these constraints is reduced to the relation between social decisions. □

Finally Rationality Constraints for Social Decision Rules are introduced.

4.2.3 Rationality constraints for Social Decision Rules

In this model individual opinions are given in the form of binary relations. In social choice theory it is convenient and illustrative to describe the binary relations under consideration by some descriptive concepts such as the following: if two options $x, y \in A$ are related by P is written as xPy and is interpreted as 'the option x is preferred to y' or 'the option x is better than the option y.' The properties of the binary relations such as acyclicity, transitivity, etc., are the core properties of interest throughout the course of this study. As will be clearer in the following subsubsection, those properties of binary relations correspond to the generally accepted ideas of rationality.

Definitions of Rationality Constraints In what follows, definitions of RCs are introduced. These are widely used properties in the literature. For a brief summary, one can be referred to (Fishburn, 1972). P is a binary relation defined on A.

1) *Irreflexivity*

If for any alternative x, xPx is never true in a binary relation P, that is no alternative is preferred to itself P is irreflexive, i.e.,

$$\forall x \in A \quad (x, x) \notin P.$$

2) *Acyclicty*

Cycles in a preference imply inconsistency in the decision making. If for any alternatives in A, $x_1Px_2P...Px_kPx_1$ is never true for any $k > 0$, that is, no alternative is preferred to another which is preferred to another (and so on) such that at some point an alternative turns out to be preferred to the very first alternative forming a cyclic preference, then P is acyclic, i.e.,

$$\rceil \exists x_1, x_2, ...x_k \in A \text{ such that } x_1Px_2P...Px_kPx_1.$$

3) *Transitivity*

For P to be transitive, for any three alternatives, if first one is preferred to the second and second one to the third, than it must be the case that first one is also preferred to the third, i.e.,

$$\forall x, y, z \in A \quad [xPy \text{ and } yPz] \Rightarrow xPz.$$

4) *Negative Transitivity*

If for any three alternatives, if first one not is preferred to the second and second one not preferred to the third, than it must be the case that first one is also not preferred to the third, i.e.,

$$\forall x, y, z \in A \quad [\, x\overline{P}y \text{ and } y\overline{P}z] \Rightarrow x\overline{P}z,$$

or equivalently,

$$\forall x, y, z \in A \quad xPy \Rightarrow [xPz \text{ or } zPy].$$

Equivalently, if P includes a pair of alternatives (xPy), then for any third additional alternative there must be a way to acouple it to this present pair, either to the top or to the bottom (xPw or wPz).

5) *Connectedness*

If any two distinct alternatives are related then P is *connected*, i.e.,

$$\forall x, y \in A, \quad x \neq y, \quad xPy \text{ or } yPx.$$

7) *Asymmetrycity*

If any two (not necessarily distinct) alternatives can not be related in both directions then P is *asymmetric*, i.e.,

$$\forall x, y \in A \quad xPy \Rightarrow y\overline{P}x.$$

Some theorems about rationality constraints (for SDRs) and their mutual relationships

Definition 7 *A binary relation is called*
(i) a strict partial order, (ii) a weak order, (iii) a linear order, if it is respectively
(i) irreflexive and transitive, (ii) irreflexive, transitive and negatively transitive, (iii) irreflexive, transitive and connected.
Below, the class of all acyclic relations are denoted by $\mathcal{AR}$ and the class of all strict partial orders are denoted by $\mathcal{SPO}$.

The interrelations between the classes follow immediately from their definitions.

Theorem 18 *(Fishburn, 1972) Acyc $\subset$ Asym.*

Theorem 19 *(Fishburn, 1972) $\mathcal{LO} \subset \mathcal{W} \subset \mathcal{SPO} \subset \mathcal{AR}$.*

Checking for a SDR for satisfying a Rationality Constraint For the SDRs that satisfy the a rationality constraint such as Asymmetricity, $\Lambda(Q_d, Asym)$ will be used. Note that when checking for connectedness the domain of definition is a profile of linear orders, otherwise weak orders. Hence whenever $(Q_d)^n = \mathcal{W}^n$ this will not be stated, like for example, $\Lambda(\mathcal{W}, Acyc) = \Lambda(Acyc)$.

Example 17 *Suppose as a rationality constraint I consider transitivity of social decision. Given a SDR, and a profile of weak orders (where each individual has transitive preferences so that $\mathcal{W} \cap Q_{\mathbf{r}} = \mathcal{W}$), does this SCP always lead to a transitive social decision given any presentation $X \in \mathcal{A}$ or is it the case that there are some situations in which the social decision is not transitive (intransitive). If there exists a profile such that a given SDR results in intransitive empty choice then this SDR does not satisfy the condition of transitivity, otherwise it does.*

Theorem 20 $[F \in \Lambda(Irref) \cap \Lambda(Trv)] \Rightarrow$
$[F \in \Lambda(Acyc) \cap \Lambda(Asym)]$.

Proof : Let $F \in \Lambda(Irref) \cap \Lambda(Trv)$. Then $\forall \overrightarrow{P} \in \mathcal{W}^n$ social decision P is a $\mathcal{SPO}$. Then P is acyclic and asymmetric by Theorem **??**. Thus $F \in \Lambda(Acyc) \cap \Lambda(Asym)$. □

5 Comparative Analysis of Social Choice Procedures

5.1 Comparative Analysis of Social Choice Correspondences

5.1.1 Normative Conditions on Coalitional Pareto Rules

In this section F_s, F_{ss} and F_w will respectively denote Strong k-majority q-Pareto rule, Strongest k-majority q-Pareto rule and Weak k-majority q-Pareto rule. Throughout the text $\mathcal{I} = \{I \subseteq N : |I| \geq k\}$ will denote the set of coalitions with cardinality greater than or equal to k. Throughout the text the following convention will be used: $X_{i,t} = \{x_i, ..., x_t\}$ and notice that $|X_{i,t}| = t - i + 1$. If $t = 0$ or $t < i$ because of values of parameters then $X_{i,t} = \emptyset$.

Theorem 21 *(Aleskerov, 1992) F_s, F_{ss} and F_w all belong to Symmetrically Central Class.*[13]

Theorem 22 $\forall \overrightarrow{P} \in \mathcal{W}^n \quad \forall X \in \mathcal{A} \quad C_s(X) \subseteq C_w(X)$ *and* $C_{ss}(X) \subseteq C_w(X)$.

Proof : Take any $\overrightarrow{P} \in \mathcal{W}^n$ and any $X \in \mathcal{A}$. Let $x \in C_s(X)$. Then $\exists I^* \in \mathcal{I}$ such that $\forall i \in I^* \quad |X \cap D_i(x)| \leq q$ hence $\left| \bigcap_{i \in I^*} X \cap D_i(x) \right| \leq q$ and $x \in C_w(X)$.

[13]In the referred paper these rules are analysed in a more general context.

Now let $x \in C_{ss}(X)$. Then $\forall I \in \mathcal{I} \quad \left|\bigcap_{i\in I} X \cap D_i(x)\right| \leq q$. Then $\exists I^* \in \mathcal{I}$ $\left|\bigcap_{i\in I^*} X \cap D_i(x)\right| \leq q$. Hence $x \in C_w(X)$.□

Theorem 23 $\forall \overrightarrow{P} \in \mathcal{W}^n \quad \forall X \in \mathcal{A} \quad k > n-k \Rightarrow C_s(X) \subseteq C_{ss}(X)$.

Proof : Let $k > n-k$ and $x \in C_s(X)$. Then $\exists I \in \mathcal{I}$ such that $\forall i \in I \quad |X \cap D_i(x)| \leq q$ where $|I| = k$. Suppose $x \notin C_{ss}(X)$. Then $\exists I' \in \mathcal{I}$ such that $\left|\bigcap_{i\in I'} X \cap D_i(x)\right| > q$ where $|I'| = k$. Then $\forall i \in I' \quad |X \cap D_i(x)| > q$ hence $I \cap I' = \emptyset$. But then $|I| + |I'| = 2k > n$. Hence $x \in C_{ss}(X)$.

Now let $k \leq n-k$. Then consider the following example.

k	$n-k$
x	$\cdot$
$\cdot$	x

In this example, $x \in C_s(X)$ but $x \notin C_{ss}(X)$ since $\exists I \in \mathcal{I}$ such that $\forall i \in I \quad |X \cap D_i(x)| = 0 \leq q$ but $\exists I' \in \mathcal{I}$ such that $\left|\bigcap_{i\in I'} X \cap D_i(x)\right| = |A| - 1 > q$.
□

Theorem 24 $k = 1 \Leftrightarrow F_s = F_w$ *and* $k = n \Leftrightarrow F_{ss} = F_w$.

Proof : Let $k = 1$. Then $\forall \overrightarrow{P} \in \mathcal{W}^n \quad \forall X \in \mathcal{A} \quad C_s(X) = \{x \in X : \exists i \in N$ such that $|X \cap D_i(x)| \leq q\} = C_w(X)$. Let $1 < k \leq n$. Then consider the following example.

1	$n-1$
x	$\cdot$
$\cdot$	x

In this example, $x \in C_w(X)$ for any $1 < k < n$ since $X \cap D_1(x) = \emptyset$ and therefore $\exists I \in \mathcal{I}$ such that $\left|\bigcap_{i\in I} X \cap D_i(x)\right| = 0$ where $1 \in I$. But $x \notin C_s(X)$ for any $1 < k \leq n$ since $\exists I' \in \mathcal{I}$ such that $\forall i \in I' \quad |X \cap D_i(x)| = |X| - 1 > q$ where $1 \notin I'$.

Now consider F_{ss}. If $k = n$ then $\mathcal{I} = \{N\}$ and therefore $C_{ss}(X) = \{x \in X : \left|\bigcap_{i\in N} X \cap D_i(x)\right| \leq q\} = C_w(X)$. For the case of $1 \leq k < n$ one can construct a counter example in the following manner.

1	$n-1$
x	$\cdot$
$\cdot$	$\cdot$
$\cdot$	x

In this example, $x \in C_w(X)$ for any $1 \leq k < n$ since $X \cap D_1(x) = \emptyset$ and therefore $\exists I \in \mathcal{I}$ such that $\left|\bigcap_{i \in I} X \cap D_i(x)\right| = 0$ where $1 \in I$. But $x \notin C_{ss}(X)$ for any $1 \leq k < n$ since $\exists I' \in \mathcal{I}$ such that $\left|\bigcap_{i \in I'} X \cap D_i(x)\right| = |X| - 1 > q$ where $1 \notin I'$. □

Theorem 25 *F_s, F_{ss} and F_w all satisfy Unanimity.*

Proof : Take any $x \in X \in \mathcal{A}$. Let $\forall i \in N \quad D_i(x) \cap X = \emptyset$. Take any k where $1 \leq k \leq n$ and constuct $\mathcal{I}$ accordingly. Then $\forall i \in I \in \mathcal{I} \quad |X \cap D_i(x)| = 0 \leq q$ and so $x \in C_s(X)$ and $\forall I \in \mathcal{I} \quad \left|\bigcap_{i \in I} X \cap D_i(x)\right| = 0 \leq q$ and so $x \in C_{ss}(X)$. Then $x \in C_w(X)$ by Theorem **??**. □

Theorem 26 *Strong n-Majority 0-Pareto rule is equal to Strongest 1-Majority 0-Pareto rule.*

Proof : Assume that the rules have the parameters stated above. Let $x \in C_s(X)$. Then $\forall i \in N \quad X \cap D_i(x) = \emptyset$ hence $x \in C_{ss}(X)$ by Theorem **??**. Let $x \notin C_s(X)$. Then $\exists i' \in N \quad X \cap D_{i'}(x) \neq \emptyset$. Then consider F_{ss}, when $k = 1$, $\exists \{i'\} \in \mathcal{I}$ $\left|\bigcap_{i \in \{i'\}} X \cap D_i(x)\right| > 0$ hence $x \notin C_{ss}(X)$. □

1) *Strong k-majority q-Pareto rule*

Theorem 27 (i) *F_s satisfies Positive non-dominance if and only if $k = 1$;*

(ii) *F_s satisfies Negative non-dominance if and only if $q = 0$;*

(iii) *If $k = 1$ then F_s does not satisfy No veto power1 but satisfies No veto power2 condition. If $1 < k \leq n$ and $q = 0$ then F_s satisfies No veto power1 (hence $F_s \in \Lambda^{NVP_2}$ by Theorem **??**). If $1 < k \leq n$ and $q > 0$ F_s does not satisfy No veto power2 condition (hence $F_s \notin \Lambda^{NVP_1}$ by Theorem **??**);*

(iv) *F_s satisfies Reinforcement Axiom if and only if $k = n$. F_s satisfies Participation Axiom if and only if $1 \leq k < n$.*

$(k = 1 \Rightarrow F_s \in \Lambda^{ND^+})$ Let $k = 1$. Then since $\overrightarrow{P} \in \mathcal{W}^n$, $\exists i_0 \in N$ such that $|X \cap D_{i_0}(x)| = 0 \leq q$ for some $x \in X$ hence $x \in C_s(X)$.

$(1 < k \leq n \Rightarrow F_s \notin \Lambda^{ND^+})$ Let $1 < k < n$. Consider the following example.

1	$n-1$
x	$\cdot$
$\cdot$	$\cdot$
$\cdot$	x

In this example, since $\forall I \in \mathcal{I} \quad \exists i \in I \quad$ such that $|X \cap D_i(x)| = |X| - 1 > q$, $x \notin C(X)$.

($q = 0 \Rightarrow F_s \in \Lambda^{ND^-}$) Take any $1 \leq k \leq n$. Assume that $\forall i \in N \quad X \cap D_i(x) \neq \emptyset$. Then since $\rceil \exists i \in N$ such that $|X \cap D_i(x)| = 0$, $\rceil \exists I \in \mathcal{I}$ such that $\forall i \in I \quad |X \cap D_i(x)| \leq 0$ hence $x \notin C(X)$.

($q \geq 1 \Rightarrow F_s \notin \Lambda^{ND^-}$) Take any $1 \leq k \leq n$. Consider the following example where $k = n \Rightarrow \forall i, j \in N \quad P_i = P_j = P_1$ where $D_1(x) = \emptyset$, $D_1(z) = \{x\}$ etc. as below.

k	$n-k$
x	y
z	z
.	.

Since $\forall i \in I \quad \forall I \in \mathcal{I} \quad |X \cap D_i(z)| = 1 \leq q$, $z \in C(X)$.

($k = 1 \Rightarrow F_s \notin \Lambda^{NVP_1}$) Consider the following example.

$n-1$	1
x	y
y	x
.	.

In this example, $y \in C(X)$ since $\exists i \in N$ such that $|X \cap D_i(y)| = 0 \leq q$.

($k = 1 \Rightarrow F_s \in \Lambda^{NVP_2}$) Assume $\forall i \in N \backslash \{j\}$ xP_iy and yP_jx. Since $\exists x_0 \in X \backslash \{y\}$ such that $|X \cap D_{i_0}(x_0)| = 0 \leq q$ for some $i_0 \in N \backslash \{j\}$, $x_0 \in C(X) \neq \{y\}$.

($1 < k < n$ and $q = 0 \Rightarrow F_s \in \Lambda^{NVP_1}$) Let $1 < k < n$ and $q = 0$. Assume $\forall i \in N \backslash \{j\}$ xP_iy and yP_jx. Take any $1 < k < n$. Then $\rceil \exists I \in \mathcal{I}$ such that $\forall i \in I \quad |X \cap D_i(y)| = 0$ hence $y \notin C(X)$. Since when $1 < k < n$ and $q = 0 \Rightarrow F_s \in \Lambda^{NVP_1}$, $F_s \in \Lambda^{NVP_2}$ by Theorem **??**.

($1 < k < n$ and $q > 0 \Rightarrow F_s \notin \Lambda^{NVP_1}$) Let $1 < k < n$ and $q > 0$. Consider the following example.

$n-1$	1
x	y
y	x
.	.

In this example, $y \in C(X)$ since $\forall i \in N \quad |X \cap D_i(y)| \leq 1 \leq q$.

(([$1 < k < n$ and $q > 0$] and $|X| \geq 3q + 2$) $\Rightarrow F_s \notin \Lambda^{NVP_2}$) Consider the following example.

$n-k$	$k-1$	1
$X_{1,q+1}$	x	$X_{2q+1,3q}$
x	y	y
y	$X_{q+2,2q}$	x
.	.	.

In this example $C(X) = \{y\}$ since $\rceil \exists I \in \mathcal{I}$ such that $\forall i \in I \quad |X \cap D_i(x)| \leq q$ for any $x \in X \backslash \{y\}$ but $\exists I \in \mathcal{I}$ such that $\forall i \in I \quad |X \cap D_i(y)| \leq q$.

($k = n$ and $q = 0 \Rightarrow F_s \in \Lambda^{NVP_1}$) Since $\forall i \in N \backslash \{j\}$ xP_iy, $\exists i \in N$ such that $|X \cap D_i(y)| > 0 = q$ hence $y \notin C(X)$. Since when $k = n$ and $q = 0 \Rightarrow F_s \in \Lambda^{NVP_1}$, $F_s \in \Lambda^{NVP_2}$ by Theorem **??**.

($k = n$ and $q > 0 \Rightarrow F_s \notin \Lambda^{NVP_1}$) Consider the following example.

$$\begin{array}{cc} \underline{n-1} & \underline{1} \\ x & y \\ y & x \\ \cdot & \cdot \end{array}$$

In this example, $y \in C(X)$ since $\forall i \in N \quad |X \cap D_i(y)| \leq 1 \leq q$.

$(([k = n$ and $q > 0]$ and $|A| \geq 2q+1) \Rightarrow F_s \notin \Lambda^{NVP_2})$ Consider the following example.

$$\begin{array}{cccccc} \underline{1} & \underline{2} & \cdot & \cdot & \underline{n-1} & \underline{n} \\ X_{1,q-1} & X_{1,q-1} & \cdot & \cdot & X_{1,q-1} & X_{q,2q-1} \\ x & x & \cdot & \cdot & x & y \\ y & y & \cdot & \cdot & y & x \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

In this example, $C(X) = \{y\}$ since $\exists i \in N$ such that $\forall z \in X \backslash \{y\} \quad |X \cap D_i(z)| > q$ but $\forall i \in N \quad |X \cap D_i(y)| \leq q$.

($1 \leq k < n \Rightarrow F_s \notin \Lambda^{Re}$) Let $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$. For the case of parameters $1 \leq k \leq n-3$ consider the following example where $I_1 = \{1, ..., k\}$ and $n_2 \geq k$.

$$\begin{array}{cc} \underline{I_1} & \underline{n_1 - k} \\ a & c \\ X\backslash\{a,b\} & X\backslash\{b,c\} \\ b & b \end{array} \qquad \begin{array}{c} \underline{n_2} \\ a, b \\ X\backslash\{a,b\} \end{array}$$

In this example, $a \in C_{N_1}(X)$ since $\forall i \in I_1 \quad |X \cap D_i(a)| = 0 \leq q$ and $a \in C_{N_2}(X)$ by $F_s \in \Lambda^U$ by Theorem **??**. Hence $a \in C_{N_1}(X) \cap C_{N_2}(X) \neq \emptyset$. Since $\forall i \in N \quad |X \cap D_i(b)| = |X| - 1 > q$, $b \notin C_{N_1}(X)$ and hence $b \notin C_{N_1}(X) \cap C_{N_2}(X)$. But since $\exists I^* \subseteq N_2 \subset N$ such that $\forall i \in I^* \quad |X \cap D_i(b)| = 0 \leq q$, hence $b \in C_N(X) \neq C_{N_1}(X) \cap C_{N_2}(X)$.

For the cases of $k = n-2$ and $k = n-1$ consider the following example. Here when $k = n-2$, $k_1 = n_1 - 2$ and $k_2 = n_2 - 2$ and respectively when $k = n-1$, $k_1 = n_1 - 1$ and $k_2 = n_2 - 1$.

$$\begin{array}{cc} \underline{k_1} & \underline{n_1 - k_1} \\ a & \cdot \\ \cdot & \cdot \\ \cdot & a \end{array} \qquad \begin{array}{cc} \underline{k_2} & \underline{n_2 - k_2} \\ a & \cdot \\ \cdot & \cdot \\ \cdot & a \end{array}$$

In this example, $a \in C_{N_1}(X) \cap C_{N_2}(X)$ but $a \notin C_N(X)$ since$\forall k \in \{n-2, n-1\}$ $(k_1 + k_2) < k$ where $n = n_1 + n_2$ and hence $\rceil \exists I \in \mathcal{I}$ such that $\forall i \in I \quad |X \cap D_i(a)| \leq q$.

($k = n \Rightarrow F_s \in \Lambda^{Re}$) Let $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$. Let $a \in C_{N_1}(X) \cap C_{N_2}(X) \neq \emptyset$. Then $\forall i \in N_1 \cup N_2 \quad |X \cap D_i(a)| \leq q$ and hence $a \in C_N(X)$. Now let $a \in C_N(A)$. Then $\forall i \in N_1 \cup N_2 \quad |X \cap D_i(a)| \leq q$ and hence $a \in C_{N_1}(X) \cap C_{N_2}(X)$.

($1 \leq k < n \Rightarrow F_s \in \Lambda^{Part}$) Let $x \in C_N(X)$. Then $\exists I^* \subseteq N$ such that $|I^*| = k$ and $\forall i \in I^* \quad |X \cap D_i(x)| \leq q$. Then since $I^* \subseteq N \cup \{j\}$, $x \in C_{N \cup \{j\}}(X)$.

($k = n$ and $|X| \geq 2q + 2 \Rightarrow F_s \notin \Lambda^{Part}$) Consider the following example.

$$\begin{array}{cccccc} \underline{P_1} & \underline{P_2} & \underline{P_3} & \cdot & \underline{P_n} & \underline{P_j} \\ X_{1,q} & X_{1,q} & X_{1,q} & \cdot & X_{q+1,2q} & X_{q+1,2q+1} \\ x & x & x & \cdot & x & x \\ . & . & . & . & . & . \end{array}$$

In this example, $C_N(X) = \{x\}$ since $\forall i \in N \quad |X \cap D_i(x)| = q \leq q$ and $\forall y \in X \backslash \{x\} \quad \exists i \in N$ such that $|X \cap D_i(x)| \geq q + 1 > q$. But $C_{N \cup \{j\}}(A) = \emptyset$ since $\forall y \in X \quad \exists i \in N$ such that $|X \cap D_i(x)| = q + 1 > q$. □

2) *Strongest k-majority q-Pareto rule*

Theorem 28 *(i) F_{ss} satisfies Positive non-dominance condition if and onlt if $k = n$;*

(ii) F_{ss} satisfies Negative non-dominance condition if and only if $k = 1$ and $q = 0$;

*(iii) If $1 \leq k < n$ and $q = 0$ then F_{ss} satisfies No veto power1 condition (hence $F_{ss} \in \Lambda^{NVP_2}$ by Theorem **??**), and if $k = n$ then F_{ss} does not satisfy No veto power1 condition. When $1 \leq k < n$ and $q > 0$ F_{ss} does not satisfy No veto power2 condition. If $k = n$ then F_{ss} satisfies No veto power2 condition;*

(iv) F_{ss} satisfies Reinforcement Axiom if and only if $k = 1$. F_{ss} satisifes Participation Axiom.

Proof : ($1 \leq k < n \Rightarrow F_{ss} \notin \Lambda^{ND^+}$) Consider the following example, the 'Voting Paradox' for the case of $k = 1$.

$$\begin{array}{ccc} \underline{P_1} & \underline{P_2} & \underline{P_3} \\ a & b & c \\ b & c & a \\ c & a & b \end{array}$$

In this example, since $\forall x \in X \quad \exists i \in N$ such that $|X \cap D_i(x)| = 0$ and $\exists \{j\} \in I$ such that $\left| \bigcap_{j \in \{j\}} X \cap D_j(x) \right| = |X| - 1 > q$, $C(X) = \emptyset$. Consider the following example for the case of $1 < k < n$.

$$\begin{array}{ccc} \underline{1} & \underline{2(k-1)} & \underline{n-2k+1} \\ x & X \backslash \{x\} & \cdot \\ y & x & \\ X \backslash \{x, y\} & & \end{array}$$

In this example $x \notin C(X)$ since $2(k - 1) \geq k$ and hence $\exists I \in \mathcal{I}$ such that $\left| \bigcap_{i \in I} X \cap D_i(x) \right| = |X| - 1 > q$.

($k = n \Rightarrow F_{ss} \in \Lambda^{ND^+}$) Since $\mathcal{I} = \{N\}$, if $|X \cap D_i(x)| = 0$ for some $i \in N$ then $\left| \bigcap_{i \in N} X \cap D_i(x) \right| = 0 \leq q$ hence $x \in C(X)$.

($k=1$ and $q=0 \Rightarrow F_{ss} \in \Lambda^{ND^-}$) Since when $k=n$ and $q=0$ $F_s \in \Lambda^{ND^-}$ by Theorem **??**, by Theorem **??** when $k=1$ and $q=0$ $F_{ss} \in \Lambda^{ND^-}$.

($k=1$ and $q>0 \Rightarrow F_{ss} \notin \Lambda^{ND^-}$) Consider the example below. In this example, $\forall i \in N \quad |X \cap D_i(y)| = 1 \leq q$ hence $y \in C(X)$.

($1<k<n \Rightarrow F_{ss} \notin \Lambda^{ND^-}$) Consider the following example.

$$\begin{array}{ccc} \underline{1} & \underline{k-1} & \underline{n-k} \\ x & z & w \\ y & y & y \\ X\backslash\{x,y\} & X\backslash\{z,y\} & X\backslash\{w,y\} \end{array}$$

In this example, since $\forall I \in \mathcal{I} \quad \left|\bigcap_{i\in I} X \cap D_i(y)\right| = 0 \leq q$, $y \in C(X)$.

($k=n \Rightarrow F_{ss} \notin \Lambda^{ND^-}$) Consider the following example.

$$\begin{array}{cc} \underline{1} & \underline{n-1} \\ x & z \\ y & y \\ . & . \end{array}$$

Since $\left|\bigcap_{i\in N} X \cap D_i(y)\right| = 0 \leq q$ and $\mathcal{I} = \{N\}$, $y \in C(X)$.

($k=1$ and $q=0 \Rightarrow F_{ss} \in \Lambda^{NVP_1}$) Since when $k=n$ and $q=0$ $F_s \in \Lambda^{NVP_1}$ by Theorem **??**, by Theorem **??** when $k=1$ and $q=0$ $F_{ss} \in \Lambda^{NVP_1}$. Since when $k=1$ and $q=0 \Rightarrow F_{ss} \in \Lambda^{NVP_1}$, $F_{ss} \in \Lambda^{NVP_2}$ by Theorem **??**.

($k=1$ and $q>0 \Rightarrow F_{ss} \notin \Lambda^{NVP_1}$) Consider the following example.

$$\begin{array}{cc} \underline{n-1} & \underline{1} \\ x & y \\ y & x \\ . & . \end{array}$$

In this example, $y \in C(X)$ since $\forall \{i\} \in \mathcal{I} \quad \left|\bigcap_{i\in \{i\}} X \cap D_i(y)\right| \leq 1 \leq q$,

(([$1 \leq k < n$ and $q>0$] and $|X| \geq 2q+1 \Rightarrow F_{ss} \notin \Lambda^{NVP_2}$) Consider the following example.

$$\begin{array}{cc} \underline{n-1} & \underline{1} \\ x & y \\ y & x_q \\ x_1 & x_{q+1} \\ . & . \\ x_{q-1} & x_{2q-1} \\ . & . \end{array}$$

In this example $\{y\} = C(X)$.

($1 < k < n$ and $q = 0 \Rightarrow F_{ss} \in \Lambda^{NVP_1}$) Assume $\forall i \in N \backslash \{j\}$ xP_iy and yP_jx. Then since $\exists I^* \in \mathcal{I}$ such that $I^* \subseteq N\backslash\{j\}$ and $\left|\bigcap_{i \in I^*} X \cap D_i(y)\right| \geq 1 > 0 = q$, $y \notin C(X)$. Since when $1 < k < n$ and $q = 0 \Rightarrow F_{ss} \in \Lambda^{NVP_1}$, $F_{ss} \in \Lambda^{NVP_2}$ by Theorem **??**.

($1 < k < n$ and $q > 0 \Rightarrow F_{ss} \notin \Lambda^{NVP_1}$) Consider the following example where $y \in C(X)$.

1	$n-1$
y	x
x	y
.	.

($k = n \Rightarrow F_{ss} \notin \Lambda^{NVP_1}$) Consider the following example where $y \in C(X)$.

$n-1$	1
x	y
.	.
.	.

($k = n \Rightarrow F_{ss} \in \Lambda^{NVP_2}$) Assume $\forall i \in N\backslash\{j\}$ xP_iy and yP_jx. Since $\exists x_0 \in X$ such that $x_0 \neq y$ and $|X \cap D_{i_0}(x_0)| = 0$ for some $i_0 \in N\backslash\{j\}$, and since when $k = n$, $F_{ss} \in \Lambda^{ND^+}$, $x_0 \in C(X) \neq \{y\}$.

($k = 1 \Rightarrow F \in \Lambda^{Re}$) Let $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$. Let $x \in C_{N_1}(X) \cap C_{N_2}(X) \neq \emptyset$. Then $\forall j \in N_1 \quad |X \cap D_j(x)| \leq q$ and $\forall t \in N_2 \quad |X \cap D_t(x)| \leq q$ and hence $\forall i \in N \qquad |X \cap D_i(x)| \leq q$. Then $x \in C_N(X)$. Now let $x \in C_N(X)$. Then $\forall i \in N \quad |X \cap D_i(x)| \leq q$. Then $\forall j \in N_1 \quad |X \cap D_j(x)| \leq q$ and $\forall t \in N_2 \quad |X \cap D_t(x)| \leq q$ and hence $x \in C_{N_1}(X) \cap C_{N_2}(X)$.

($1 < k < n \Rightarrow F_{ss} \notin \Lambda^{Re}$) Let $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$. Consider the following example where $n_i - \lceil k/2 \rceil \geq k$ where $i \in \{1, 2\}$.

$n_1 - \lceil k/2 \rceil$	$\lceil k/2 \rceil$		$\lceil k/2 \rceil$	$n_2 - \lceil k/2 \rceil$
a	$X\backslash\{a\}$		$X\backslash\{a\}$	a
.	a		a	.
.				.

In this example, $a \in C_{N_1}(X)$ since $\forall I \in \mathcal{I}_1 \quad \left|\bigcap_{i \in I} X \cap D_i(a)\right| = 0 \leq q$ and $a \in C_{N_2}(X)$ since $\forall I \in \mathcal{I}_2 \quad \left|\bigcap_{i \in I} X \cap D_i(a)\right| = 0 \leq q$ and hence $a \in C_{N_1}(X) \cap C_{N_2}(X) \neq \emptyset$. But since $2\lceil k/2 \rceil \geq k$, $\exists I \in \mathcal{I}$ such that $\left|\bigcap_{i \in I_1} X \cap D_i(a)\right| = |X| - 1 > q$ and hence $a \notin C_N(X)$.

($k = n \Rightarrow F_{ss} \notin \Lambda^{Re}$) Let $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$. Consider the following example.

$$\begin{array}{cc} \underline{1} & \underline{n_1 - 1} \\ X\backslash\{a\} & b \\ a & X\backslash\{a,b\} \\ & a \end{array} \qquad \begin{array}{cc} \underline{1} & \underline{n_2 - 1} \\ a & b \\ X\backslash\{a\} & a \\ & X\backslash\{a,b\} \end{array}$$

In this example, $b \in C_{N_1}(X) \cap C_{N_2}(X) \neq \emptyset$ since $F_{ss} \in \Lambda^{ND^+}$ by Theorem **??**. Similarly, $a \in C_N(X)$ but $a \notin C_{N_1}(X)$ since $\left|\bigcap_{i\in N_1} X \cap D_i(a)\right| = |X| - 1 > q$.

($F_{ss} \in \Lambda^{Part}$) Let $x \in C_N(X)$. Then $\forall I \in \mathcal{I}_N$ $\left|\bigcap_{i\in I} X \cap D_i(x)\right| \leq q$. Suppose $x \notin C_{N\cup\{j\}}(X)$. Then $\exists I^* \in \mathcal{I}_{N\cup\{j\}}$ such that $j \in I^*$ and $\left|\bigcap_{i\in I^*} X \cap D_i(x)\right| > q$. But $\left|\bigcap_{i\in I^*} X \cap D_i(x)\right| = \left|(\bigcap_{i\in I} X \cap D_i(x)) \cap (\bigcap_{i\in\{j\}} X \cap D_i(x))\right| \leq q$ for some $I \in \mathcal{I}_N$. So there is a contradiction and hence $x \in C_{N\cup\{j\}}(X)$. $\square$

3) *Weak k-majority q-Pareto rule*

Theorem 29 *(i) F_w satisfies Positive non-dominance;*

(ii) F_w satisfies Negative non-dominance condition if and only if $k = 1$ and $q = 0$;

(iii) F_w satisfies No veto power2 condition but it does not satisfy No veto power1 condition;

(iv) F_w does not satisfy Reinforcement and F_w satisfies Participation Axioms.

($F_w \in \Lambda^{ND^+}$) Let $k = 1$. Since $F_s \in \Lambda^{ND^+}$ when $k = 1$,by Theorem **??**, $F_w \in \Lambda^{ND^+}$. Let $1 < k < n$. Since if $\exists i \in N$ $|X \cap D_i(x)| = 0 \leq q$ then $\exists I \in \mathcal{I}$ such that $\left|\bigcap_{i\in I} X \cap D_i(x)\right| = 0 \leq q$ where $i \in I$ and hence $x \in C(X)$. Let $k = n$. Since $F_{ss} \in \Lambda^{ND^+}$ when $k = n$,by Theorem **??**, $F_w \in \Lambda^{ND^+}$.

($k = 1$ and $q = 0 \Rightarrow F_w \in \Lambda^{ND^-}$) Since when $k = 1$ and $q = 0$, $F_s \in \Lambda^{ND^-}$, by Theorem **??**, $F_w \in \Lambda^{ND^-}$.

($k = 1$ and $q > 0 \Rightarrow F_w \notin \Lambda^{ND^-}$) Since when $k = 1$ and $q > 0$, $F_s \notin \Lambda^{ND^-}$, by Theorem **??**, $F_w \notin \Lambda^{ND^-}$.

($1 < k < n$ and $q = 0 \Rightarrow F_w \notin \Lambda^{ND^-}$) Since when $1 < k < n$ and $q = 0$, $F_{ss} \notin \Lambda^{ND^-}$ and by Theorem **??** $C_{ss}(X) \subseteq C_w(X)$, $F_w \notin \Lambda^{ND^-}$.

($1 \leq k < n$ and $q > 0 \Rightarrow F_w \notin \Lambda^{ND^-}$) Since when $1 < k < n$ and $q > 0$, $F_{ss}, F_s \notin \Lambda^{ND^-}$, and by Theorem **??** $C_{ss}(X) \subseteq C_w(X)$ and $C_s(X) \subseteq C_w(X)$, $F_w \notin \Lambda^{ND^-}$.

($k = n \Rightarrow F_w \notin \Lambda^{ND^-}$) Since when $k = n$, $F_{ss} \notin \Lambda^{ND^-}$, by Theorem **??**, $F_w \notin \Lambda^{ND^-}$.

($k = 1 \Rightarrow F_w \notin \Lambda^{NVP_1}$) Since when $k = 1$, $F_s \notin \Lambda^{NVP_1}$, by Theorem **??**, $F_w \notin \Lambda^{NVP_1}$.

($k = 1 \Rightarrow F_w \in \Lambda^{NVP_2}$) Since when $k = 1$, $F_s \in \Lambda^{NVP_2}$, by Theorem **??**, $F_w \in \Lambda^{NVP_2}$.

($1 < k < n$ and $q = 0 \Rightarrow F_w \notin \Lambda^{NVP_1}$) Consider the following example.

$$\begin{array}{cc} \underline{n-1} & \underline{1} \\ x & y \\ \cdot & \cdot \\ \cdot & \cdot \end{array}$$

In this example, $y \in C(X)$ since $F_w \in \Lambda^{ND^+}$ by Theorem **??**.

($1 < k < n$ and $q > 0 \Rightarrow F_w \notin \Lambda^{NVP_1}$) Since when $1 < k < n$ and $q > 0 \Rightarrow F_s \notin \Lambda^{NVP_1}$ and by Theorem **??** $C_s(X) \subseteq C_w(X)$, $F_w \notin \Lambda^{NVP_1}$.

($1 < k < n \Rightarrow F_w \in \Lambda^{NVP_2}$) Assume $\forall i \in N\backslash\{j\}$ xP_iy and yP_jx. Since $\exists x_0 \in X$ such that $x_0 \neq y$ and $|X \cap D_{i_0}(x_0)| = 0 \leq q$ for some $i_0 \in N$, and since $F_w \in \Lambda^{ND^+}$, $x_0 \in C(X) \neq \{y\}$.

($k = n \Rightarrow F_w \notin \Lambda^{NVP_1}$) Since when $k = n$, $F_{ss} \notin \Lambda^{NVP_1}$, by Theorem **??**, $F_w \notin \Lambda^{NVP_1}$.

($k = n \Rightarrow F_w \in \Lambda^{NVP_2}$) Since when $k = 1$, $F_{ss} \in \Lambda^{NVP_2}$, by Theorem **??**, $F_w \in \Lambda^{NVP_2}$.

($F_w \notin \Lambda^{Re}$) Let $N_1 \cup N_2 = N$ and $N_1 \cap N_2 = \emptyset$. Consider the following example.

$$\begin{array}{cccc} \underline{k} & \underline{n_1 - k} & \underline{1} & \underline{n_2 - 1} \\ b & \cdot & a & b \\ X\backslash\{a,b\} & \cdot & b & \cdot \\ a & a & \cdot & \cdot \end{array}$$

In this example, $b \in C_{N_1}(X) \cap C_{N_2}(X) \neq \emptyset$ since $F_w \in \Lambda^{ND^+}$ by Theorem **??**. But $a \in C_N(X)$ since $F_w \in \Lambda^{ND^+}$ by Theorem **??** and $a \notin C_{N_1}(X)$ since $\forall I \in \mathcal{I}_1$ $\left|\bigcap_{i \in I} X \cap D_i(a)\right| = |X| - 1 > q$.

($F_w \in \Lambda^{Part}$) Let $x \in C_N(X)$. Then $\exists I^* \in \mathcal{I}_N$ $\left|\bigcap_{i \in I} X \cap D_i(x)\right| \leq q$. Then $\exists I^* \in \mathcal{I}_{N \cup \{j\}}$ such that $\left|\bigcap_{i \in I} X \cap D_i(x)\right| \leq q$. Hence $x \in C_{N \cup \{j\}}(X)$. □

5.1.2 Rationality constraints on CPRs

General Results on Coalitional Pareto Rules and Rationality Constraints

Theorem 30 $\forall F \in \{F_s, F_{ss}, F_w\}$ $\quad F \in \Lambda^{ND^+} \Rightarrow F \in \Lambda(NE)$.

Proof : Let $F \in \Lambda^{ND^+}$. Then if $\exists i \in N$ such that $|X \cap D_i(x)| = 0$ for some $x \in X$, $x \in C(X)$. But $\forall \overrightarrow{P} \in \mathcal{W}^n$ $\quad \exists P_i \in \overrightarrow{P}$ such that $|X \cap D_i(x)| = 0$ for some $x \in X$ hence $x \in C(X) \neq \emptyset$. □

Theorem 31 *(Aleskerov, 1992) F_s, F_{ss} and F_w are rules*[14] *that belong to $\Lambda(H)$.*

Theorem 32 *When $0 < q < |A| - 1$, F_s, F_{ss} and F_w are rules that do not belong to $\Lambda(Con^+)$.*

Proof : Let $\forall i \in N \quad \forall y \in A\backslash\{x\} \quad yP_ix$. Let $0 < q < |A| - 1$. Consider F_s. Since $\forall i \in N \quad |A \cap D_i(x)| = |A| - 1 > q$, $\rceil\exists I \in \mathcal{I}$ such that $\forall i \in I \quad |A \cap D_i(x)| \leq q$ for any $1 \leq k \leq n$, hence $x \notin C_s(A)$. But $\forall y \in A\backslash\{x\} \quad x \in C(\{x, y\})$ since $\forall i \in N$ $|\{x, y\} \cap D_i(x)| = 1 \leq q$.

Consider F_w and F_{ss}. Since $\forall i \in N \quad |A \cap D_i(x)| = |A| - 1$, $\forall I \in \mathcal{I}$ $\left|\bigcap_{i\in I} A \cap D_i(x)\right| = |A\backslash\{x\}| = |A| - 1 > q$ for any $1 \leq k \leq n$, hence $x \notin C_w(A)$ and hence $x \notin C_{ss}(A)$ by Theorem **??**. But $\forall y \in A\backslash\{x\} \quad x \in C_{ss}(\{x, y\})$ since $\left|\bigcap_{i\in I}\{x, y\} \cap D_i(x)\right| = 1 \leq q$ and hence $\forall y \in A \quad x \in C_w(\{x, y\})$ by Theorem **??**.

When $0 < q < |A| - 1$, since $F_s, F_w, F_{ss} \notin \Lambda(Con^+)$, $F_s, F_w, F_{ss} \notin \Lambda(C)$. $\square$

Theorem 33 *(Aleskerov, 1992) Concordance condition is satisfied by a rule belonging to Symmetrically Central Class if and only if it is Strong n-Majority 0-Pareto rule (or equivalently Strongest 1-Majority 0-Pareto rule)*[15] *or Strongest k-Majority 0-Pareto rule where $1 < k < n$.*

Proof : The proof of the statement for F_s when $k = n$ and $q = 0$ can be found in (Aleskerov,1992). The latter result is not found in the original theorem therefore its proof follows. Consider F_{ss} and let $q = 0$ and $1 < k < n$. Let $x \in C(X_1) \cap C(X_2)$. Then $\rceil\exists y_1 \in X_1$ such that $|V(y_1, x; \{P_i\})| \geq k$ and $\rceil\exists y_2 \in X_2$ such that $|V(y_2, x; \{P_i\})| \geq k$.Then $\forall y \in X_1 \cup X_2 \quad |V(y, x; \{P_i\})| < k$. Hence $\forall I \in \mathcal{I}$ $\left|\bigcap_{i\in I}(X_1 \cup X_2) \cap D_i(x)\right| = 0 \leq q = 0$. Hence $x \in C(X_1 \cup X_2)$. $\square$

Remark 13 : Direct Condorcet condition is satisfied by a rule belonging to set $\{F_s, F_w, F_{ss}\}$ if and only if it satisfies Concordance condition since all of these rules belong to $\Lambda(H)$ and $\Lambda(H) \cap \Lambda(Con^+) \subset \Lambda(C)$.

Theorem 34 *(Aleskerov, 1992) In Symmetrically Central Class, no rule satisfies Arrow 's Choice Axiom and only Weak k-majority q-Pareto satisfies Independence of Outcast condition.*

Remark 14 : Since I have $PI = H \cap O$ by Theorem **??**, from the Symmetrically Central Class Path Independence condition is satisfied by only Weak k-majority q-Pareto rule.

[14] In the referred paper these rules are analysed in a more general context.

[15] In the referred paper, another equivalent form of the rule, 'Condorcet Operator' is stated as the only rule satisfying Concordance.

Remark 15 : By Theorem **??** F_s satisfies Path Independence condition when $k = 1$ and F_{ss} satisfies Path Independence condition when $k = n$.

Remark 16 : By Theorem **??** since $F_s, F_{ss}, F_w \notin \Lambda(ACA)$ but $F_s, F_{ss}, F_w \in \Lambda(H)$, $F_s, F_{ss}, F_w \notin \Lambda(ACA^*)$ and $F_s, F_{ss}, F_w \notin \Lambda(H^-)$. $\square$

Theorem 35 $\forall F \in \Lambda(H) \quad F \in \Lambda(NE) \Leftrightarrow F \in \Lambda(FP)$.

Proof : Take any $F \in \Lambda(H)$. Assume $F \in \Lambda(NE)$. Then $\forall \overrightarrow{P} \in \mathcal{W}^n \quad C(\cdot) \in NE \cap H = H^+ \subset FP$ and hence $F \in \Lambda(FP)$. Now assume $F \in \Lambda(FP)$. Since $\forall \overrightarrow{P} \in \mathcal{W}^n \quad C(\cdot) \in FP \subset NE$, $F \in \Lambda(NE)$. $\square$

Remark 17 : $\forall F \in \{F_s, F_{ss}, F_w\} \quad F \in \Lambda(H)$ by Theorem **??**.

1) *Strong k-majority q-Pareto rule*

Theorem 36 *(i)* $F_s \in \Lambda(NE) \Leftrightarrow |A| \geq \lceil n/(k-1) \rceil (q+1)$ *and* $F_s \in \Lambda(NE) \Leftrightarrow F_s \in \Lambda(FP)$ *by Theorem* **??***;*

(ii) F_s *satisfies Weak non-resoluteness condition if and only if* $k = n$ *and* $q = 0$*;*

(iii) F_s *satisfies Weak Axiom of Revealed Preferences if and only if* $k = n$ *and* $q = 0$.

Proof : ($F_s \in \Lambda(NE) \Leftrightarrow |A| \geq \lceil n/(k-1) \rceil (q+1)$) First assume $F_s \notin \Lambda(NE)$. Then $\exists \overrightarrow{P} \in \mathcal{W}^n$ such that $C(A) = \emptyset$. Denote $n^+_{j+1}(x, \overrightarrow{P}) = card\{i \in N : |A \cap D_i(x)| \leq j\}$. Since $C(A) = \emptyset$, $\forall x \in A \quad n^+_{q+1}(x, \overrightarrow{P}) < k$. By $\overrightarrow{P} \in \mathcal{W}^n$, $\sum_{x \in A} n^+_{q+1}(x, \overrightarrow{P}) = n(q+1) \leq |A|(k-1)$. Hence $|A| \geq \lceil n/(k-1) \rceil (q+1)$. Now assume that $|A| \geq \lceil n/(k-1) \rceil (q+1)$. First consider the following example for the case of $n = 5$ and $k = 3$ hence $\lceil n/(k-1) \rceil = 3$. Here $|A| \geq 3q + 3$.

$\underline{2}$	$\underline{2}$	$\underline{1}$
$X_{1,q+1}$	$X_{q+2,2(q+1)}$	$X_{2(q+1)+1,3(q+1)}$
.	.	.

In this example, $\forall x \in A \quad n^+_{q+1}(x, \overrightarrow{P}) < k = 3$ hence $C(A) = \emptyset$. Note that N is partitioned into $\lceil n/(k-1) \rceil = 3$ coalitions, where all but one coalition have cardinality $k - 1 = 2$. The remaining coalition has cardinality $n - (\lceil n/(k-1) \rceil - 1)(k-1) = 5 - (3-1)(3-1) = 1$.

To generalize this example to the case of $1 < k \leq n$ apply the following procedure. Start with partitioning N into coalitions with cardinality $k-1$ as long as this is possible. If n is not divisible by $k - 1$ then there will remain one coalition with cardinality $n - (\lceil n/(k-1) \rceil - 1)(k-1) < k - 1$. Then construct the following structure in which there are $\lceil n/(k-1)\rceil = s$ coalitions with their respective cardinalities $k_1 = k_2 = ... = k_{s-1} = k-1$and $k_s = n - (\lceil n/(k-1)\rceil - 1)(k-1)$.

$$\begin{array}{ccc} \underline{k_1} & \underline{k_2} & \underline{k_s} \\ X_{1,\ q+1} & X_{q+2,\ 2(q+1)} & X_{(s-1)(q+1)+1,\ s(q+1)} \\ \cdot & \cdot & \cdot \end{array}$$

In this example, $C(A) = \emptyset$ since voters of any two different coalitions have non-intersecting top $q+1$ alternatives and no single coalition has cardinality greater than or equal to k.

For the case of $k = 1$ since $\forall \overrightarrow{P} \in \mathcal{W}^n$, $\exists x \in A$ such that $n^+(x, \overrightarrow{P}) \geq k = 1$ whenever $|A| < \infty$, hence $x \in C(A) \neq \emptyset$.

($1 \leq k < n \Rightarrow F_s \notin \Lambda(WNR)$) For the case of $1 \leq k < n$ and $q > 0$ consider the following example.

$$\begin{array}{cc} \underline{1} & \underline{n-1} \\ A\backslash\{x\} & x \\ x & A\backslash\{x\} \end{array}$$

In this example $C(A) = A$. For the case of $1 \leq k < n-1$ and $q = 0$ consider the following example where $n - k \geq k$.

$$\begin{array}{cc} \underline{k} & \underline{n-k} \\ x & A\backslash\{x\} \\ A\backslash\{x\} & x \end{array}$$

In this example $C(A) = A$. For the case of $k = n-1$ and $q = 0$ consider the following example where $n = 3$.

$$\begin{array}{ccc} \underline{P_1} & \underline{P_2} & \underline{P_3} \\ x, y & y, z & x, z \\ z & x & y \end{array}$$

In this example $C(A) = A$.

($k = n$ and $q > 0 \Rightarrow F_s \notin \Lambda(WNR)$) For the case of $k = n$ and $q > 0$ consider the following example.

$$\begin{array}{cccc} \underline{P_1} & \underline{P_2} & \cdot & \underline{P_n} \\ x & x & \cdot & x \\ A\backslash\{x\} & A\backslash\{x\} & \cdot & A\backslash\{x\} \end{array}$$

In this example $C(A) = A$.

($k = n$ and $q = 0 \Rightarrow F_s \in \Lambda(WNR)$) Since by assumption $\forall i \in N \quad P_i \neq \emptyset$, $\exists i \in N$ such that $\exists x \in A$ such that $|A \cap D_i(x)| \neq \emptyset$ hence $x \in C(A)$.

(([$k = 1$ and $q \geq 0$] and $|A| \geq q+4$) $\Rightarrow$ ($F_s \notin \Lambda(WARP)$)) Consider the example below. In this example, $x \in C(X_1)$ and $z \notin C(X_1)$ but $x \notin C(X_2)$ and $z \in C(X_2)$.

$(([1<k<n-1$ and $q>0]$ and $|A|\geq q+4) \Rightarrow (F_s \notin \Lambda(WARP))$ Consider the following example where $n-k\geq k$, $X_1=\{x,y,z\}\cup X_{1,q}$ and $X_2=\{x,w,z\}\cup X_{1,q}$.

$\underline{k}$	$\underline{n-k}$
$X_{1,q}$	y
w	$X_{1,q}$
x	z
y	w
z	x
.	.

In this example, $x\in C(X_1)$ and $z\notin C(X_1)$ but $x\notin C(X_2)$ and $z\in C(X_2)$.

$(([1<k<n-1$ and $q=0]$ and $|A|\geq 3) \Rightarrow (F_s \notin \Lambda(WARP))$ Consider the following example.

$\underline{k-1}$	$\underline{k-1}$	$\underline{k-1}$
x	y	z
y	z	x
z	x	y
.	.	.

In this example, $C(\{x,y\})=\{x\}$, and $C(\{y,z\})=\{y\}$ but $C(\{x,z\})=\{z\}$.

$((k=n-1$ and $|A|\geq q+4) \Rightarrow (F_s \notin \Lambda(WARP))$ Consider the following example for the case of $q=0$ where $X_1=\{x,y,z\}$ and $X_2=\{x,w,z\}$.

$\underline{1}$	$\underline{n-2}$	$\underline{1}$
w	x,y,z	y
x	.	z
.	.	x
.	.	.

In this example, $x\in C(X_1)$ and $z\notin C(X_1)$ but $x\notin C(X_2)$ and $z\in C(X_2)$.

For the case of $k=1$, consider the following example where $X_{1,0}=\emptyset$, $X_1=\{x,y,z\}\cup X_{1,q}$ and $X_2=\{x,w\}\cup X_{1,q}$.

$\underline{1}$	$\underline{n-2}$	$\underline{1}$
x,y	w	y,z,w
$X_{1,q-1}$	$X_{1,q-1}$	$X_{1,q-1}$
x_q	x_q	x_q
.	x	x
.	z	.
.	.	.

In this example, $x\in C(X_1)$ and $x_q\notin C(X_1)$ but $x\notin C(X_2)$ and $x_q\in C(X_2)$.

$((k=n$ and $q>0) \Rightarrow (F_s \notin \Lambda(WARP))$ Consider the following example. Here $|A|\geq 2q+2$, $X_1=\{x\}\cup X_{1,q+1}$ and $X_2=\{x\}\cup X_{1,q}\cup X_{q+2,2q+1}$.

$$\begin{array}{cc} \underline{1} & \underline{n-1} \\ X_{1,q+1} & x \\ x & X_{q+2,2q+1} \\ X_{q+2,2q+1} & X_{1,q+1} \\ . & . \end{array}$$

In this example, $x_1 \in C(X_1)$ and $x \notin C(X_1)$ but $x \in C(X_2)$ and $x_1 \notin C(X_2)$.

($k = n$ and $q = 0 \Rightarrow F_s \in \Lambda(WARP)$) Let $k = n$ and $q = 0$. Suppose that $\exists x_1, ..., x_m$ such that

$$\begin{array}{c} x_1 \in C(X_1) \text{ and } x_2 \in X_1 \backslash C(X_1) \\ x_2 \in C(X_2) \text{ and } x_3 \in X_2 \backslash C(X_2) \\ . \\ . \\ x_m \in C(X_m) \text{ and } x_1 \in X_m \backslash C(X_m). \end{array}$$

When $k = n$ and $q = 0$, whenever $x \in C(X)$ then $\forall y \in X \quad \rceil \exists i \in N$ such that yP_ix. Futhermore $z \notin C(X) \Rightarrow \exists i_0 \in N$ such that $\exists y \in X$ such that $yP_{i_0}z$ and, by $\overrightarrow{P} \in \mathcal{W}^n$ and $x \in C(X)$, $xP_{i_0}z$. Hence $x_j \in C(X_j)$ and $x_{j+1} \in X_j \backslash C(X_j) \Rightarrow x_j P_{i_j} x_{j+1}$ for some $i_j \in N$. Hence $x_1 P_{i_1} x_2$. Now, consider P_{i_1} and x_3. Since $x_3 \notin C(X_2)$, $x_3 \overline{P}_{i_1} x_2$ and since $P_{i_1} \in \mathcal{W}$, $x_1 P_{i_1} x_3$ and similarly $x_1 P_{i_1} x_4, ..., x_1 P_{i_1} x_{m-1}$ and $x_1 P_{i_1} x_m$ the last of which contradicts $x_m \in C(X_m)$ and $x_1 \in X_m \backslash C(X_m)$. □

2) *Strongest k-majority q-Pareto rule*

Theorem 37 *(i)* $F_{ss} \in \Lambda(NE) \Leftrightarrow |A| < (q+1)\nu(\mathcal{I})$ *where* $\nu(\mathcal{I})$ *is the Nakamura number*[16] *associated to family* $\mathcal{I}$ *(Aleskerov, 1996)*[17] *and* $F_{ss} \in \Lambda(NE) \Leftrightarrow F_{ss} \in \Lambda(FP)$ *by Theorem* **??**;

(ii) F_{ss} *satisfies Weak non-resoluteness condition if and only if* $k = 1$ *and* $q = 0$;

(iii) F_{ss} *satisfies Weak Axiom of Revealed Preferences if and only if* $k = 1$ *and* $q = 0$.

Proof : ($k = 1$ and $q = 0 \Rightarrow F_{ss} \in \Lambda(WNR)$) Since when $k = n$ and $q = 0$ $F_s \in \Lambda(WNR)$, by Theorem **??** when $k = 1$ and $q = 0$ $F_{ss} \in \Lambda(WNR)$.

($k = 1$ and $q > 0 \Rightarrow F_{ss} \notin \Lambda(WNR)$) Consider the following example.

$$\begin{array}{cc} \underline{1} & \underline{n-1} \\ x & y \\ A \backslash \{x\} & A \backslash \{y\} \end{array}$$

Since $\forall i \in N \quad \forall a \in A \quad |A \cap D_i(a)| \leq 1 \leq q$, $C(A) = A$.

[16] This number shows the smallest number of sets in $\mathcal{I}$ with empty intersection. Whenever $\bigcap_{I \in \mathcal{I}} I \neq \emptyset$ then by definition $\nu(\mathcal{I}) = \infty$.

In the referred paper the problem of Non-emptiness is investigated in a more general context.

[17] In the referred paper the problem of Non-emptiness is investigated in a more general context.

(($1 < k \leq n$ and $|A| \geq 3 \Rightarrow F_{ss} \notin \Lambda(WNR)$) For the case of $1 < k < n-1$, consider the following example.

$$\begin{array}{ccc} \underline{k-1} & \underline{k-1} & \underline{k-1} \\ x & y & z \\ X\backslash\{x\} & X\backslash\{y\} & X\backslash\{z\} \\ . & . & . \end{array}$$

In this example, $C(A) = A$. For the case of $k = n-1$, consider the following example.

$$\begin{array}{ccc} \underline{1} & \underline{n-2} & \underline{1} \\ x & y & z \\ X\backslash\{x\} & X\backslash\{y\} & X\backslash\{z\} \\ . & . & . \end{array}$$

In this example, $C(A) = A$.

Since when $k = n$ by Theorem **??** $C_s(A) \subseteq C_{ss}(A)$ and for $k = n$ $F_s \notin \Lambda(WNR)$, $F_{ss}(A) \notin \Lambda(WNR)$.

(($k = 1$ and $q = 0 \Rightarrow F_{ss} \in \Lambda(WARP)$) Since when $k = n$ and $q = 0$ $F_s \in \Lambda(WARP)$, by Theorem **??** when $k = 1$ and $q = 0$ $F_{ss} \in \Lambda(WARP)$.

(([$k = 1$ and $q > 0$] and $|A| \geq 2q + 2) \Rightarrow F_{ss} \notin \Lambda(WARP)$) Consider the following example where $X_1 = \{x, y\} \cup X_{1,q}$ and $X_2 = \{x, y\} \cup X$.

$$\begin{array}{ccc} \underline{1} & \underline{1} & \underline{1} \\ x & y & y \\ X_{q+1,2q} & X_{1,q} & x \\ y & x & X_{1,q} \\ . & . & . \end{array}$$

In this example, $y \in C(X_1)$ and $x \notin C(X_1)$ but $y \notin C(X_2)$ and $x \in C(X_2)$.

(($1 < k < n-1$ and $|A| \geq 2q + 3) \Rightarrow F_{ss} \notin \Lambda(WARP)$) Consider the following example where $X_1 = \{x, y\} \cup X_{1,q}$ and $X_2 = \{x, y\} \cup X_{q+1,2q+1}$.

$$\begin{array}{ccc} \underline{k-1} & \underline{k-1} & \underline{k-1} \\ x & y & X_{q+1,2q+1} \\ X_{1,q} & X_{q+1,2q+1} & x \\ y & x & X_{1,q} \\ . & . & y \\ . & & . \end{array}$$

In this example, $x \in C(X_1)$ and $y \notin C(X_1)$ but $y \in C(X_2)$ and $x \notin C(X_2)$.

(([$k = n-1$ and $q > 0$] and $|A| \geq 2q + 3) \Rightarrow F_{ss} \notin \Lambda(WARP)$) Consider the following example where $X_1 = \{x, y\} \cup X_{1,q+1}$ and $X_2 = \{x, y\} \cup X_{q+2,2q+2}$.

$$\begin{array}{ccc} \underline{1} & \underline{n-2} & \underline{1} \\ x & X_{1,q+1} & y \\ X_{1,q+1} & y & X_{q+1,2q+1} \\ y & X_{q+2,2q+2} & x \\ . & x & . \\ . & . & . \end{array}$$

In this example, $x \in C(X_1)$ and $y \notin C(X_1)$ but $y \in C(X_2)$ and $x \notin C(X_2)$.

(([$k = n-1$ and $q = 0$] and $|A| \geq 4$) $\Rightarrow F_{ss} \notin \Lambda(WARP)$) Consider the following example where $X_1 = \{x, y, z\}$ and $X_2 = \{x, y, w\}$.

$\underline{1}$	$\underline{n-2}$	$\underline{1}$
x	w	z
w	y	x, y
y	z	$\cdot$
$\cdot$	x	$\cdot$
$\cdot$	$\cdot$	$\cdot$

In this example, $x \notin C(X_1)$ and $y \in C(X_1)$ but $y \notin C(X_2)$ and $x \in C(X_2)$.

(($k = n$ and $|A| \geq 2q + 4$) $\Rightarrow F_{ss} \notin \Lambda(WARP)$) Consider the following example where $X_1 = \{x, y\} \cup X_{1,q+1}$ and $X_2 = \{x, y\} \cup X_{q+2,2q+2}$.

$\underline{1}$	$\underline{n-1}$
$X_{q+2,2q+2}$	$X_{1,q+1}$
x	y
$X_{1,q+1}$	$X_{q+2,2q+2}$
y	x
$\cdot$	$\cdot$

In this example, $x \in C(X_1)$ and $y \notin C(X_1)$ but $y \in C(X_2)$ and $x \notin C(X_2)$. □

3) *Weak k-majority q-Pareto rule*

Theorem 38 *(i) F_w satisfies Non-emptiness hence Fixed point condition;*

(ii) F_w does not satisfy Weak non-resoluteness condition and Weak Axiom of Revealed Preferences.

($F_w \in \Lambda(NE)$) Since when $k = 1$ $F_s \in \Lambda(NE)$, by Theorem **??** $F_w \in \Lambda(NE)$.

For the case of $1 < k < n$ consider any $P_{i_0} \in \overrightarrow{P} \in \mathcal{W}^n$. Then $\exists i_0 \in N$ such that $|X \cap D_{i_0}(x)| = 0$ for some $x \in X$ hence $\exists I \in \mathcal{I}$ such that $i_0 \in I$ and therefore $\left| \bigcap_{i \in I} X \cap D_i(x) \right| = 0$. Thus $x \in C(X) \neq \emptyset$. Hence $F_w \in \Lambda(NE)$.

When $k = n$, $F_w = F_{ss}$ by Theorem **??**. Then $F_{ss} \in \Lambda(NE) \Leftrightarrow |A| < (q+1)\nu(\mathcal{I})$ where $\mathcal{I} = \{N\}$. But $\nu(\mathcal{I}) = \infty$ since $\bigcap_{I \in \mathcal{I}} I = N \neq \emptyset$. Then since $|A| < \infty$ always holds, when $k = n$, $F_{ss} = F_w \in \Lambda(NE)$.

Since $F_w \in \Lambda(NE)$ by Theorem **??**, $F_w \in \Lambda(FP)$.

($F_w \notin \Lambda(WNR)$) Since when $k = 1$ $F_s \notin \Lambda(WNR)$, by Theorem **??** $F_w \notin \Lambda(WNR)$.

For the case of $1 < k < n$, consider the following example.

$\underline{1}$	$\underline{n-1}$
x	$A \backslash \{x\}$
$A \backslash \{x\}$	x

Since $\forall x \in A \quad \exists i \in N$ such that $|A \cap D_i(x)| = 0$ and $F_w \in \Lambda(ND^+)$ by Theorem **??** $C(A) = A$.

Since when $k = n$ $F_{ss} \notin \Lambda(WNR)$, by Theorem **??** $F_w \notin \Lambda(WNR)$.

($k = 1 \Rightarrow F_w \notin \Lambda(WARP)$) When $k = 1$, since $F_s \notin \Lambda(WARP)$, $F_w \notin \Lambda(WARP)$ and when $k = n$, since $F_{ss} \notin \Lambda(WARP)$, $F_w \notin \Lambda(WARP)$ by Theorem **??**.

(($1 < k < n$ and $|A| \geq 2q+4$) $\Rightarrow F_w \notin \Lambda(WARP)$) Consider the following example where $X_1 = \{x, z\} \cup X_{1,q+1}$ and $X_2 = \{x, z\} \cup X_{q+2,2q+2}$. Let $n - k + 1 = 2(k-1) \geq k$.

$k-1$	$n-k+1$
$X_{q+2,2q+2}$	$X_{1,q+1} y$
x	z
$X_{1,q+1}$	$X_{q+2,2q+2}$
z	x
.	.

In this example, $x \in C(X_1)$ and $z \notin C(X_1)$ but $z \in C(X_2)$ and $x \notin C(X_2)$. □

5.1.3 Normative conditions on Positional SCCs

Theorem 39 *(Young, 1975) (i) All scoring voting correspondences (choosing the subset of candidates with highest total score) satisfy Reinforcement axiom. (ii) There is no Condorcet consistent*[18] *voting correspondence satisfying Reinforcement axiom.*

Theorem 40 *(Moulin, 1986) (i) All scoring voting rules satisfy Participation axiom. (ii) If $|A| \geq 4$ then there is no Condorcet consistent rule satisfying Participation axiom.*

4) *Plurality*

Theorem 41 *Plurality satisfies Unanimity, Negative non-dominance, No veto power1 (hence $F \in \Lambda^{NVP_2}$ by Theorem* **??***) conditions and, Reinforcement and Participation Axioms, it does not satisfy Locality (hence Monotonicity and Neutrality2) and Positive non-dominance conditions.*

Proof : ($F \in \Lambda^U$) Take any $x \in X \in \mathcal{A}$. Let $\forall i \in N \quad D_i(x) \cap X = \emptyset$. Then $\forall i \in N \quad \rceil \exists y \in X$ such that yP_ix, then $\forall y \in X \quad n^+(x, \overrightarrow{P}) = n \geq n^+(y, \overrightarrow{P})$, then $x \in C(X)$. Since $F \in \Lambda^U$, then $F \in \Lambda^{NI^+}$ by Theorem **??**.

($F \notin \Lambda^{ND^+}$) Consider the following example.

1	2	3
x, y	y	y
.	.	.

In this example, $\exists i = 1 \in N$ such that $X \cap D_i(x) = \emptyset$ but $x \notin C(X) = \{y\}$.

[18] A Condorcet consistent rule is a rule which chooses the Condorcet Winner (CW) when it exists. A Condorcet Winner is an alternative that is preferred by a majority (not necessarily consisting the same voters) to every other alternative.

($F \in \Lambda^{ND^-}$) Take any $x \in X \in \mathcal{A}$. Let $\forall i \in N \quad X \cap D_i(x) \neq \emptyset$. Then $\forall i \in N \quad \exists y_i \in X$ such that $y_i P_i x$, then $\exists y_i \in X \quad n^+(x, \overrightarrow{P}/X) = 0 < n^+(y_i, \overrightarrow{P}/X)$, then $x \notin C(X)$. Since $F \in \Lambda^{ND^-}$, then $F \in \Lambda^{NI^-}$ by Theorem **??**.

Remark 18: If F does not satisfy Locality, than it does not satisfy Monotonicity and Neutrality2. The example given for the Borda procedure (the next rule that is analyzed) in the case of Locality will also work for Plurality. For illustrative purposes I provided separate example for Monotonicity and Neutrality2 below.

($F \notin \Lambda^{M}$) Consider the following example.

$\overrightarrow{P}/X$: 1	2	3	4	5	$\overrightarrow{P'}/X$: 1	2	3	4	5
x	y	x	z	w	x	z	x	z	z
y, z	z	$\cdot$	$\cdot$	z	$\cdot$	x	$\cdot$	x	w
w	w	$\cdot$	$\cdot$	x	$\cdot$	$\cdot$	$\cdot$	$\cdot$	x
	x	$\cdot$	x	y	$\cdot$	$\cdot$	$\cdot$	$\cdot$	y

In this example, $n^+(x, \overrightarrow{P}/X) = 2 > n^+(y, \overrightarrow{P}/X) = n^+(z, \overrightarrow{P}/X) = n^+(w, \overrightarrow{P}/X) = 1$ and therefore $x \in C(X)$. But, although $\forall i \in N \quad X \cap D'_i(x) \subseteq X \cap D_i(x)$, $x \notin C'(X) = \{z\}$.

($F \notin \Lambda^{Ne_2}$) Consider the following example.

$\overrightarrow{P}/X$: 1	2	3	$\overrightarrow{P'}/X$: 1	2	3
x	x	z	y, z	y, z	z
y, z	y, z	x, y			y

In this example, $C(A) = \{x\}$. Take $X = \{x, y, z\}$ and $X' = \{y, z\}$. Although $\forall i \in N \quad X \cap D_i(x) = X' \cap D_i(y)$, $y \notin C(X') = \{z\}$.

($F \in \Lambda^{NVP_1}$) Let $\forall i \in N \backslash \{j\} \quad x P_i y$ and $y P_j x$. Then since $\overrightarrow{P} \in \mathcal{W}^n$ and $n \geq 3$, $n^+(x, \overrightarrow{P}/X) > n^+(y, \overrightarrow{P}/X)$ hence $y \notin C(X)$. Since $F \in \Lambda^{NVP_1}$, $F \in \Lambda^{NVP_2}$ by Theorem **??**.

($F \in \Lambda^{RA}$) Since Plurality rule is a scoring rule, then $F \in \Lambda^{RA}$ by Theorem **??**.

($F \in \Lambda^{PA}$) Since Plurality rule is a scoring rule $F \in \Lambda^{PA}$ by Theorem **??**. $\square$

5) *Inverse Plurality*

Theorem 42 $\mathbf{F} \in \mathbf{\Lambda}(\overline{\mathbf{U}}) \cap \mathbf{\Lambda}(\overline{\mathbf{ND^-}}) \cap \mathbf{\Lambda}(\overline{\mathbf{M}}) \cap \mathbf{\Lambda}(\overline{\mathbf{Ne}}) \cap \mathbf{\Lambda}(\mathbf{An}) \cap \mathbf{\Lambda}(\overline{\mathbf{NVP}}) \cap \mathbf{\Lambda}(\mathbf{RA}) \cap \mathbf{\Lambda}(\mathbf{PA})$

Proof: ($F \in \Lambda(U)$) Take any $x \in X \in \mathcal{A}$. Let $\forall i \in N \quad D_i(x) \cap X = \emptyset$. Suppose $x \notin C(X)$. Since $F \in \Lambda(NE)$ (for proof of this fact consult next section), $\exists y \in X$ such that $y \in C(X)$. Then $n^-(y, \overrightarrow{P}) < n^-(x, \overrightarrow{P})$ hence $\exists i_0 \in N$ such that $y P_{i_0} x$ which contradicts the assumption. So $x \in C(X)$. Since $F \in \Lambda(U)$, then $F \in \Lambda(NI^+)$ by Theorem **??**.

($F \notin \Lambda(NVP2)$) Consider the following example where $A = \{x, y\}$ and $\overrightarrow{P}$ is a weak order which is not a linear order[19].

$$\overrightarrow{P}$$

$\underline{1}$	$\underline{2}$	$\underline{3}$
x, y	x, y	y
		x

In this example, $\forall i \in N \backslash \{3\} \quad D_i(x) \cap A = \emptyset$ but $x \notin C(A) = \{y\}$. Since $F \notin \Lambda(NVP2)$, then $F \notin \Lambda(NVP1)$ by Theorem **??**. Also, that $F \notin \Lambda(ND^+)$ can be shown via this example.

($F \notin \Lambda(ND^-)$) Consider the following example.

$$\overrightarrow{P}$$

$\underline{1}$	$\underline{2}$	$\underline{3}$
x	y	z
w	w	w
y, z	z, x	x, y

In this example, $n^-(w, \overrightarrow{P}) = 0 < n^-(x, \overrightarrow{P}) = n^-(y, \overrightarrow{P}) = n^-(z, \overrightarrow{P}) = 2$ hence $C(A) = \{w\}$ where $\forall i \in N \quad D_i(w) \cap X \neq \emptyset$.

($F \in \Lambda(NI^-)$). Consider the following example which can be applied to any $\{x, y\} \subseteq X \in \mathcal{A}$ where $x \neq y$ with $n \geq 2$.

$$\overrightarrow{P}$$

$\underline{1}$	$\underline{2}$	$\underline{:}$	$\underline{:}$	$\underline{n}$
X	X	X	X	$X \backslash \{x\}$
				x

In this example, $\forall y \in X \backslash \{x\} \quad n^-(y, \overrightarrow{P}) = n - 1 < n^-(x, \overrightarrow{P}) = n$ implying $x \notin C(X)$.

($F \notin \Lambda(M)$) Consider the following example.

$\overrightarrow{P}$

$\underline{1}$	$\underline{2}$	$\underline{3}$
x	x	y
y	y	x

$\overrightarrow{P'}$

$\underline{1}$	$\underline{2}$	$\underline{3}$
x, y	x, y	y
		x

[19] For the case of linear orders, although all the indifference classes consist of singletons including indifference class of alternatives getting 'top vote', $F \notin \Lambda(NVP2)$. Consider the following example where $\forall i \in N \backslash \{3\} \quad D_i(x) \cap X = \emptyset$ but $x \notin C(X) = \{z\}$.

$\overrightarrow{P}$

$\underline{1}$	$\underline{2}$	$\underline{3}$
x	x	y
z	z	z
y	y	x

In this example, although $\forall i \in N \quad D'_i(x) \cap X \subseteq D_i(x) \cap X$ and $x \in C(\{x, y\})$, $x \notin C'(\{x, y\}) = \{y\}$.

($F \notin \Lambda(Ne)$) Consider the following example where $A = \{x, y, z\}$.

$$\begin{array}{ccc} & \overrightarrow{P} & \\ \underline{1} & \underline{2} & \underline{3} \\ x & x & z \\ y,z & y,z & x,y \end{array} \qquad \begin{array}{ccc} & \overrightarrow{P} & \\ \underline{1} & \underline{2} & \underline{3} \\ y,z & y,z & z \\ & & y \end{array}$$

In this example, $C(A) = \{x\}$. Take $X = \{x, y, z\} = A$ and $X' = \{y, z\}$. Although $\forall i \in N \quad D_i(x) \cap X = D_i(y) \cap X'$, $y \notin C(X') = \{z\}$ violating neutrality.

($F \in \Lambda(An)$) Let a bijection $\eta : N \to N$ be given. Then $n^-(x, \overrightarrow{P}) = n^-_\eta(x, \overrightarrow{P'})$ where $\overrightarrow{P'} = \{P_{\eta(i)}\}$. So $x \in C(X) \Leftrightarrow x \in C_\eta(X)$.

($F \in \Lambda(RA)$) Since Invese Plurality rule is a scoring rule (Moulin, 1988), then $F \in \Lambda(RA)$ by Theorem **??**.

($F \in \Lambda(PA)$) Since Inverse Plurality rule is a scoring rule (Moulin, 1988), then for $|A| \geq 4$, $F \in \Lambda(PA)$ by Theorem **??**.

6) *Borda*

Theorem 43 *Borda rule satisfies Unanimity condition and Reinforcement and Participation Axioms, it does not satisfy Locality (hence Monotonicity and Neutrality2), Positive and Negative non-dominance and No veto power2 (hence $F \notin \Lambda^{NVP_1}$ by Theorem* **??***) conditions.*

Proof : ($F \in \Lambda^U$) Take any $x \in X \in \mathcal{A}$. Let $\forall i \in N \quad D_i(x) \cap X = \emptyset$. Then $\forall i \in N \quad \forall y \in A \quad \forall X \in \mathcal{A} \quad L_i(x) \cap X \supseteq L_i(y) \cap X$ and hence $\forall i \in N \quad \forall y \in A \quad \forall X \in \mathcal{A} \quad B_i(x) \geq B_i(y)$ and therefore $B(x) \geq B(y)$. Hence $x \in C(X)$.

($F \notin \Lambda^L$) Consider the following example.

$$\begin{array}{ccc} & \overrightarrow{P}/X & \\ \underline{1} & \underline{2} & \underline{3} \\ x & x & y \\ y & y & x \\ \cdot & \cdot & \cdot \end{array} \qquad \begin{array}{ccc} & \overrightarrow{P'}/X & \\ \underline{1} & \underline{2} & \underline{3} \\ x,y & x,y & y \\ \cdot & \cdot & x \\ \cdot & \cdot & \cdot \end{array}$$

In this example, although $\forall i \in N \quad D'_i(x) \cap X = D_i(x) \cap X$ and $\{x\} = C(X)$, $x \notin C'(X) = \{y\}$ hence F does not satisfy locality and hence $F \notin \Lambda^{Ne_2}$. and $F \notin \Lambda^M$.

($F \notin \Lambda^{ND^+}$) Consider the example below. There $X \cap D_1(x) = \emptyset$ but $x \notin C(X)$.

($F \notin \Lambda^{ND^-}$) Consider the following example.

$$\begin{array}{ccc} \underline{1} & \underline{1} & \underline{1} \\ x & y & z \\ w & w & w \\ y,z & z,x & x,y \end{array}$$

In this example, $B(w) = 6 > B(x) = B(y) = B(z) = 3$ hence $C(A) = \{w\}$ although $\forall i \in N \quad D_i(w) \cap X \neq \emptyset$.

($|A| > |N| \Rightarrow F \notin \Lambda^{NVP_2}$) Consider the following example.

$\underline{n-1}$	$\underline{1}$
$X\backslash\{y\}$	y
y	$X\backslash\{y\}$

In this example, $\forall a \in A\backslash\{y\} \quad B(a) = |N| - 1$ and $B(y) = |A| - 1$. Hence if $|A| > |N|$ then $C(X) = \{y\}$. Then if $|A| > |N|$ then $F \notin \Lambda^{NVP_1}$ by Theorem **??**.

($F \in \Lambda^{RA}$) Since Borda rule is a scoring rule, then $F \in \Lambda^{RA}$ by Theorem **??**.

($F \in \Lambda^{PA}$) Since Borda rule is a scoring rule $F \in \Lambda^{PA}$ by Theorem **??**. □

7) *Inverse Borda*

Theorem 44 *Inverse Borda rule satisfies Unanimity. It does not satisfy Locality, Positive and Negative non-dominance and No veto power2 conditions and Reinforcement and Participation Axioms.*

Proof : ($F \in \Lambda^U$) Take any $x \in X \in \mathcal{A}$. Let $\forall i \in N \quad D_i(x) \cap X = \emptyset$. Then $\forall i \in N \quad L_i(x) \cap X \supseteq L_i(y) \cap X$ and hence $\forall i \in N \quad B_i(x) \geq B_i(y)$ for any $y \in X$ and therefore $B(x) \geq B(y)$. So x can not be eliminated from X. Assume that $z \in X$ is eliminated from X and the resulting set is $X' \subset X$. Since $\forall i \in N \quad D_i(x) \cap X' = \emptyset$ for any $X' \subseteq X$, x will not be eliminated from any X'. Hence $x \in C(X)$. Since $F \in \Lambda^U$, then $F \in \Lambda^{NI^+}$ by Theorem **??**.

($F \notin \Lambda^M$) Consider the following example.

$\overrightarrow{P}$			$\overrightarrow{P'}$		
$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{1}$	$\underline{2}$	$\underline{3}$
x	w	x	x, w	w	x, y, z, w
y	x	w, y, z	y	x, y, z	
z	y, z		z		
w					

In this example, although $\forall i \in N \quad D'_i(x) \cap X = D_i(x) \cap X$ and $x \in C(X) = \{x\}$, $x \notin C'(X) = \{w\}$. Hence F is not local. Then $F \notin \Lambda^M$ and $F \notin \Lambda^{Ne_2}$ by Theorem **??**.

($F \in \Lambda^{Ne_1}$) Let $\sigma : A \to A$ be a bijection and $\forall i \in N \quad \forall x \in A \quad \sigma(L_i(x)) = L_i(\sigma(x))$ and $\sigma(D_i(x)) = D_i(\sigma(x))$. Let $\sigma(P_i) = \{(\sigma(x), \sigma(y)) \in A \times A : (x, y) \in A \times A\}$ and $\sigma(\overrightarrow{P})$ be the corresponding profile. Then $\forall X \in \mathcal{A} \quad B(x, \overrightarrow{P}/X) \leq B(y, \overrightarrow{P}/X)$ $\Leftrightarrow B(\sigma(x), \sigma(\overrightarrow{P})/\sigma(X)) \leq B(\sigma(y), \sigma(\overrightarrow{P})/\sigma(X))$ hence $x \in C(X) \Leftrightarrow \sigma(x) \in C'(\sigma(X))$ where $C'(\sigma(X)) = F(\sigma(X), \sigma(\overrightarrow{P})/\sigma(X))$.

($F \in \Lambda^{An}$) Let $\eta : N \to N$ be a bijection. Take any $X \in \mathcal{A}$ and any $\overrightarrow{P} \in \mathcal{W}$. Consider the contraction $\overrightarrow{P}/X$. Then $\forall X \in \mathcal{A} \quad B(x, \overrightarrow{P}/X) = B(x, \overrightarrow{P'}/X)$ where $\overrightarrow{P'}/X = \{P_{\eta(i)}/X\}_{\eta(i) \in N}$. So $\forall X \in \mathcal{A} \quad x \in C(X) \Leftrightarrow x \in C'(X)$.

($F \notin \Lambda^{ND^+}$) Consider the example below. There $x \notin C(A)$ although $X \cap D_1(x) = \emptyset$.

($F \notin \Lambda^{ND^-}$) Consider the following example.

$$\begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} \\ x & z & w & x & w \\ y & y & y & y & y \\ z,w & w,x & z & w,z & x,z \\ & & x & & \end{array}$$

In this example, $C(A) = \{y\}$ where $\forall i \in N \quad D_i(y) \cap X \neq \emptyset$.

($|X| > |N| \Rightarrow F \notin \Lambda^{NVP_2}$) Consider the following example where $|X| > |N|$.

$$\begin{array}{cc} \underline{n-1} & \underline{1} \\ X \backslash \{y\} & y \\ y & X \backslash \{y\} \end{array}$$

In this example, $\forall a \in X \backslash \{y\} \quad B(a) = |N| - 1$ and $B(y) = |X| - 1$. Hence if $|X| > |N|$ then $C(X) = \{y\}$. Then if $|X| > |N|$ then $F \notin \Lambda^{NVP_1}$ by Theorem **??**.

($F \notin \Lambda^{RA}$ and $F \notin \Lambda^{PA}$) First following lemmas are introduced.

Lemma 1 $\forall \overrightarrow{P} \in \mathcal{W} \quad \forall X \in \mathcal{A}$ *if* $a \in A$ *is a Condorcet Winner (CW) on* $\overrightarrow{P}/X$ *then* $B(a, \overrightarrow{P}/X) \geq \lceil n/2 \rceil (m-1)$ *where* $m = |X|$.

Proof : Let a be a CW on $\overrightarrow{P}/X$. Then each $b \in X \backslash \{a\}$ exists at least $\lceil n/2 \rceil$ times in lower contour sets of a, i.e., $\forall b \in X \backslash \{a\} \quad \left| V(a, b; \overrightarrow{P}/X) \right| \geq \lceil n/2 \rceil$. Hence $B(a, \overrightarrow{P}/X) \geq \lceil n/2 \rceil (m-1)$ where $m = |X|$. □

Lemma 2 *Inverse Borda rule is Condorcet consistent.*

Proof : Suppose $a \in A$ is a CW on $\overrightarrow{P}/A$ and it is eliminated from some $X \in \mathcal{A}$. Then $\forall x \in X \backslash \{a\} \quad B(x, \overrightarrow{P}/X) \geq B(a, \overrightarrow{P}/X)$ and $\exists x_0 \in X \backslash \{a\}$ such that $B(x_0, \overrightarrow{P}/X) > B(a, \overrightarrow{P}/X)$. Then sum of the Borda Scores for $\overrightarrow{P}/X$ is strictly greater than $|X|$ times $B(a)$, i.e., $\sum_{y \in X} B(y) > mB(a)$ where $m = |X|$. But this is infeasible because it exceeds the total of Borda Scores for a linear order on X in which case this total is the greatest, i.e., $\sum_{y \in X} B(y) > mB(a) \geq m(m-1) \lceil n/2 \rceil \geq m(m-1)(n/2)$ by Lemma **??** and definition of $\lceil \cdot \rceil$, but $\sum_{y \in X} B(y) > m(m-1)(n/2) = \max_{\overrightarrow{P} \in \mathcal{W}} \sum_{z \in X} B(z, \overrightarrow{P}/X)$. So a is never eliminated, that is a is always chosen by Inverse Borda rule. □

($F \notin \Lambda^{RA}$) Since Inverse Borda rule is Condorcet Consistent by Lemma **??**, then $F \notin \Lambda^{RA}$ by Theorem **??**.

($F \notin \Lambda^{PA}$) Since Inverse Borda rule is Condorcet Consistent by Lemma **??**, then whenever $|A| \geq 4$, $F \notin \Lambda^{PA}$ by Theorem **??**. □

5.1.4 Rationality Constraints for Positional SCCs

4) *Plurality*

Theorem 45 *Plurality satisfies Non-emptiness condition but it does not satisfy Weak non-resoluteness, Inverse and Direct Condorcet, Independence of Outcast, Dual heritage, Fixed point conditions and Weak axiom of revealed preferences.*

Proof : ($F \in \Lambda(NE)$) Take any $X \in \mathcal{A}$ and $\overrightarrow{P} \in \mathcal{W}$ and consider the contraction $\overrightarrow{P}/X$. Then $\exists x \in X$ such that $n^+(x, \overrightarrow{P}/X) \geq 0$. Now, consider the set of scores of all such alternatives $N^+ = \{n^+(x, \overrightarrow{P}/X) \in Z^+ : x \in X\}$ where Z^+ is the set of all nonnegative integers. Since maximum of N^+ exists, the social decision is never empty, i.e.,

$$\arg \max_{n^+(x,\overrightarrow{P}/X) \in N^+} n^+(x, \overrightarrow{P}/X) \in C(X) \neq \emptyset.$$

($F \notin \Lambda(WNR)$) Consider the following example where $A = \{x, y, z\}$.

1	1	1
x, y	y, z	x, z
z	x	y

In this example, $C(A) = A \not\subset A$ hence $F \notin \Lambda(WNR)$.
($F \notin \Lambda(Con^-)$) Consider the following example.

3	2	2
x	y	z
y	z, x	y
z		x

In this example, $x \in C(A)$ but $x \notin C(\{x, y\})$. Since $F \notin \Lambda(Con^-)$, then $F \notin \Lambda(H)$ and $F \notin \Lambda(ACA)$, since $F \notin \Lambda(H)$, $F \notin \Lambda(ACA^*)$, and $F \notin \Lambda(PI)$ by Theorem **??**.

($F \notin \Lambda(Con^+)$) Consider the previous example. In this example, $y \in C(\{x, y\}) \cap C(\{y, z\})$ but $y \notin C(A)$. Since $F \notin \Lambda(Con^+)$, then $F \notin \Lambda(C)$ by Theorem **??**.

($F \notin \Lambda(O)$) Consider the previous example. In this example, $C(A) = \{x\} \subset \{x, y\} \subset A$ but $C(A) = \{x\} \neq \{y\} = C(\{x, y\})$.

($F \notin \Lambda(H^-)$) Consider the previous example. In this example, $\{x, y\} \subset A$ but $C(\{x, y\}) = \{y\} \nsubseteq \{x\} = C(A)$.

($F \notin \Lambda(FP)$) Consider the previous example. Take $X' = \{a, b\}$ and $X = A$. Since $C(A) = \{x\}$, $\rceil \exists a \in \{x, y\} \subset A$ such that $a \in C(\{x, y\}) = \{y\}$.

($F \notin \Lambda(WARP)$) Consider the previous example. Since $C(A) = \{x\}$, xGy and xGz holds.But since $C(\{x, y\}) = \{y\}$, yGx also holds. $\square$

5) *Inverse Plurality*

Theorem 46 *Inverse Plurality satisfies Non-emptiness but it does not satisfy Weak non-resoluteness, Inverse and Direct Condorcet Principles, Independence of Outcast, Dual heritage, Fixed point conditions and Weak axiom of revealed preferences.*

Proof: ($F \in \Lambda(NE)$) Take any $X \in \mathcal{A}$ and $\overrightarrow{P} \in \mathcal{W}$. Consider the contraction $\overrightarrow{P}/X$. $\exists x \in X$ such that $n^-(x, \overrightarrow{P}/X) > 0$. Now, consider the set of scores of all such alternatives $N^- = \{n^-(x, \overrightarrow{P}/X) \in Z^+ : x \in X\}$ where Z^+ is the set of all nonnegative integers. Since maximum of N^- exists, the social decision is never empty, i.e., $\arg \max_{n^-(x,\overrightarrow{P}/X) \in N^-} n^-(x, \overrightarrow{P}/X) \in C(X) \neq \emptyset$.

($F \notin \Lambda(WNR)$) Consider the following example where $A = \{x, y, z\}$.

$$\begin{array}{ccc} & \overrightarrow{P} & \\ \underline{1} & \underline{2} & \underline{3} \\ y,z & x,z & x,y \\ x & y & z \end{array}$$

In this example, $\forall a \in A \quad n^-(a, \overrightarrow{P}) = 1$, hence $C(A) = A \not\subset A$ violating weak non-resoluteness .

($F \notin \Lambda(Con^-)$) Consider the following example where $A = \{x, y, z\}$ and $N = \{1, 2, ..., 7\}$.

$$\begin{array}{ccc} & \overrightarrow{P} & \\ \underline{3} & \underline{2} & \underline{2} \\ y,z & x & x \\ x & z & y \\ & y & z \end{array}$$

In this example, $y \in C(A) = \{y, z\}$ but $y \notin C(\{x, y\})$ violating Con^-. Since $F \notin \Lambda(Con^-)$, then $F \notin \Lambda(H)$ and $F \notin \Lambda(ACA)$ by Theorems **??** and **??**. Since $F \notin \Lambda(H)$, then $F \notin \Lambda(ACA^*)$ by Theorem **??** and $F \notin \Lambda(PI)$ by Theorem **??**.

($F \notin \Lambda(Con^+)$) Consider the previous example. In this example, $x \in C(\{x, y\}) \cap C(\{x, z\})$ but $x \notin C(A)$ violating Con^+. Since $F \notin \Lambda(Con^+)$, then $F \notin \Lambda(C)$ by Theorem **??**.

($F \notin \Lambda(O)$) Consider the following example.

$$\begin{array}{ccc} & \overrightarrow{P} & \\ \underline{3} & \underline{2} & \underline{2} \\ x & z,w & w \\ y & x & z \\ z,w & y & y \\ & & x \end{array}$$

In this example, $\{x, y\} = C(A) \subseteq \{x, y\} \subset A$ but $C(A) = \{x, y\} \neq \{x\} = C(\{x, y\})$.

($F \notin \Lambda(H^-)$) Consider the following example.

$$\begin{array}{ccc} & \overrightarrow{P} & \\ \underline{3} & \underline{2} & \underline{2} \\ y,z & x & x \\ x & z & y \\ & y & z \end{array}$$

In this example, $\{x,y\} \subset A$ but $C(\{x,y\}) = \{x\} \not\subset \{y,z\} = C(A)$.

($F \notin \Lambda(FP)$) Consider the previous example. In this example, $\rceil\exists a \in A$ such that $a \in X \subseteq A \Rightarrow a \in C(X)$ since $\{y,z\} = C(A)$ but $y \notin C(\{x,y\})$ and $z \notin C(\{x,z\})$.

($F \notin \Lambda(WARP)$) Consider the previous example. In this example, $C(A) = \{y,z\} \Rightarrow yGx$ and zGx. However, $C(\{x,y\}) = \{x\} \Rightarrow xGy$ violating acyclicity of G.

6) *Borda*

Theorem 47 *Borda satisfies Non-emptiness condition but it does not satisfy Weak non-resoluteness, Inverse and Direct Condorcet, Independence of Outcast, Dual heritage, Fixed point conditions and Weak axiom of revealed preferences.*

Proof : ($F \in \Lambda(NE)$) Take any $X \in \mathcal{A}$ and $\overrightarrow{P} \in \mathcal{W}$ and consider the contraction $\overrightarrow{P}/X$. Then $\exists x \in X$ such that $B(x, \overrightarrow{P}/X) \geq 0$. Now, consider the set of scores of all such alternatives $B = \{B(x, \overrightarrow{P}/X) \in Z^+ : x \in X\}$ where Z^+ is the set of all nonnegative integers. Since maximum of B exists, the social decision is never empty, i.e., $\arg \max_{B(x,\overrightarrow{P}/X) \in B} B(x, \overrightarrow{P}/X) \in C(X) \neq \emptyset$.

($F \notin \Lambda(WNR)$) Consider the following example where $A = \{x,y,z,w\}$.

1	1	1	1
x	w	w,z	y
y	z	x,y	x
z,w	y,x		z
			w

In this example, $C(A) = A \not\subset A$ since $B(x) = B(y) = B(z) = B(w) = 5$.

($F \notin \Lambda(Con^-)$) Consider the following example.

1	1	1
x	x	y
y	y	z,w
z,w	z,w	x

In this example, $C(A) = \{y\}$ but $y \notin C(\{x,y\})$ violating Con^-. Since $F \notin \Lambda(Con^-)$, then $F \notin \Lambda(H)$ and $F \notin \Lambda(ACA)$, since $F \notin \Lambda(H)$, $F \notin \Lambda(ACA^*)$ and $F \notin \Lambda(PI)$ by Theorem **??**.

($F \notin \Lambda(Con^+)$) Consider the previous example. In this example, $x \in C(\{x,y\}) \cap C(\{x,z\}) \cap C(\{x,w\})$ but $x \notin C(A)$. Since $F \notin \Lambda(Con^+)$, then $F \notin \Lambda(C)$ by Theorem **??**.

($F \notin \Lambda(O)$) Consider the previous example. In this example, $C(A) = \{y\} \subset \{x,y\} \subset A$ but $C(A) = \{y\} \neq \{x\} = C(\{x,y\})$.

($F \notin \Lambda(H^-)$) Consider the previous example. In this example, $\{x,y\} \subset A$ but $C(\{x,y\}) = \{x\} \not\subset \{y\} = C(A)$.

($F \notin \Lambda(FP)$) Consider the previous example. In this example, $\rceil \exists a \in A$ such that $a \in X \subseteq A \Rightarrow a \in C(X)$ since $\{y\} = C(A)$ but $y \notin C(\{x, y\})$ where $X = \{x, y\}$.

($F \notin \Lambda(WARP)$) Consider the previous example. In this example, $C(A) = \{y\} \Rightarrow yGx$, yGz and yGw. However, $C(\{x, y\}) = \{x\} \Rightarrow xGy$. $\square$

7) *Inverse Borda*

Theorem 48 *Inverse Borda satisfies Non-emptiness and Direct Condorcet Principle but it does not satisfy Weak non-resoluteness, Inverse Condorcet, Independence of Outcast, Concordance, Dual heritage, Fixed point conditions and Weak axiom of revealed preferences.*

Proof: ($F \in \Lambda(NE)$) Take any $X \in \mathcal{A}$ and $\overrightarrow{P} \in \mathcal{W}$. Consider the contraction $\overrightarrow{P}/X$. Since Borda Rule is never empty, there will always be some alternative which is not eliminated from any $X' \subseteq X$.

($F \notin \Lambda(WNR)$) Consider the following example where $A = \{x, y, z, w\}$.

1	1	1
x	y	z
y, z	x, z	x, y

In this example, $C(A) = A \not\subset A$ since $B(x) = B(y) = B(z) = 2$.

($F \notin \Lambda(Con^-)$) Consider the following example.

1	1	1
x, y	w	z
z	x	w
w	y	x
	z	y

In this example, $z \in C(A) = \{x, z, w\}$ but $z \notin C(\{y, z\})$. Since $F \notin \Lambda(Con^-)$, then $F \notin \Lambda(H)$ and $F \notin \Lambda(ACA)$ by Theorems **??** and **??**. Since $F \notin \Lambda(H)$, then $F \notin \Lambda(ACA^*)$ by Theorem **??** and $F \notin \Lambda(PI)$ by Theorem **??**.

($F \in \Lambda(Con^+)$) Take any $x \in X \in \mathcal{A}$. Let $x \in C(\{x, y\})$ for each $y \in X$. Then $\forall y \in X \quad B(x, \overrightarrow{P}/\{x, y\}) \geq B(y, \overrightarrow{P}/\{x, y\})$. Then$\forall y \in X \quad |\{i \in N : xP_iy\}| \geq |\{i \in N : yP_ix\}|$. Suppose $\forall i \in N \quad \forall y \in X \quad x\overline{P}_iy$ and $y\overline{P}_ix$. Then $\forall i \in N \quad \forall y, z \in X \quad y\overline{P}_iz$ and $z\overline{P}_iy$ since $\forall P_i \in \overrightarrow{P} \quad P_i \in \mathcal{W}$ which further implies $x \in C(X) = X$. Then $\exists y \in X \quad |\{i \in N : xP_iy\}| > |\{i \in N : yP_ix\}|$ and $\forall y \in X \quad |\{i \in N : xP_iy\}| \geq |\{i \in N : yP_ix\}|$.

Consider the case of linear orders. Then $\forall X \in \mathcal{A} \quad B(x, \overrightarrow{P}/X) \geq (|X| - 1) \lceil n/2 \rceil$. Suppose $\forall y \in X \backslash \{x, z\} \quad B(y, \overrightarrow{P}/X) \geq B(z, \overrightarrow{P}/X) > B(x, \overrightarrow{P}/X)$ so that x is eliminated. This implies total of score to be greater than $|X|$ times score of x, i.e., $\sum_{y \in X} B(y, \overrightarrow{P}/X) > (|X|)(|X| - 1) \lceil n/2 \rceil \geq (|X|)(|X| - 1)(n/2)$. But we know that for linear orders total of scores must equal to $(|X|)(|X| - 1)(n/2)$ which leads to the following contradicition. $\sum_{y \in X} B(y, \overrightarrow{P}/X) > (|X|)(|X| - 1)(n/2) = \sum_{a \in X} B(a, \overrightarrow{P}/X)$.

($F \notin \Lambda(C)$) Consider the following example.[20]

$\overrightarrow{P}$

1	2	3	4	5	6
x	y	w	x	y	z
w	x	z	z	x	w
z	z	y	w	w	y
y	w	x	y	z	x

From $\{x, y, z, w\}$ only x is selected. But $C(x, y, w) = \{x, y, w\}$ and $C(x, y, z) = \{x, y, z\}, violating Concordance.$

($F \notin \Lambda(O)$) Consider the following example.

$\overrightarrow{P}$

1	2	3	4	5	6
x	y	y	y	x	z, w
y	z	z	z, w	w, z	x
z, w	x	x	x	y	y
	w	w			

In this example, $C(A) = \{y\} \subset \{x, y\} \subset A$ but $C(A) = \{y\} \neq \{x, y\} = C(\{x, y\})$.

($F \notin \Lambda(H^-)$) Consider the example below. In this example, $\{y, z\} \subset A$ but $C(\{y, z\}) = \{y\} \not\subset \{x\} = C(A)$.

($F \notin \Lambda(FP)$) Consider the following example.

1	1	1
x, y	w	z
z	x	w
w	y	x
	z	y

In this example, $\rceil \exists a \in A$ such that $a \in X \subseteq A \Rightarrow a \in C(X)$ since $C(A) = \{x, z, w\}$ but $C(\{x, y\}) = \{y\}$.

($F \notin \Lambda(WARP)$) Consider the previous example. In this example, $C(A) = \{x, z, w\}$ hence zGy. However, $C(\{y, z\}) = \{y\}$ hence yGz. $\square$

5.2 Comparative Analysis of FVRs

5.2.1 Normative conditions on FVRs

8) *Approval Voting*

Theorem 49 *Approval Voting satisfies Positive and Negative Pareto conditions and Reinforcement Axiom. It does not satisfy Locality condition. It satisfies No veto power condition.*

[20] I thank Borghan N. Narajabad for providing this example.

Proof : ($F \in \Lambda^{U^+}$) Take any $x \in X \in \mathcal{A}$. Let profile $\{C_i(\cdot)\}$ be such that $\forall i \in N \quad x \in C_i(X)$. Since $|V(x, X; \{C_i(\cdot)\})| = n \geq |V(y, X; \{C_i(\cdot)\})|$ for any $y \in X$, $x \in C(X)$.

($F \in \Lambda^{U^-}$) Let $x, y \in X$ where $x \neq y$. Let $|V(x, X; \{C_i(\cdot)\})| = 0$ and $|V(y, X; \{C_i(\cdot)\})| \neq 0$. Then $x \notin C(X)$. Now, let $X = \{x\}$. Let $|V(x, X; \{C_i(\cdot)\})| = 0$. Then $x \notin C(X)$ satisfying U^-.

($F \notin \Lambda^M$) Let the following two profiles $\{C_i(\cdot)\}$ and $\{\widetilde{C}_i(\cdot)\}$ and $A = \{x, y, z\}$ be given.

X	$C_1(A)$	$C_2(A)$	$C_3(A)$	$C(A)$
A	$\{x\}$	$\{y\}$	$\{z\}$	$\{x, y, z\}$

X	$\tilde{C}_1(A)$	$\tilde{C}_2(A)$	$\tilde{C}_3(A)$	$\tilde{C}(A)$
A	$\{x, y\}$	$\{y\}$	$\{y\}$	$\{y\}$

Since $\forall a, b \in A \quad |V(a, A; \{C_i(\cdot)\})| = |V(b, A; \{C_i(\cdot)\})| = 1$, $A = C(A)$. One can check that $V(x, A; \{C_i(\cdot)\}) = V(x, A; \{\widetilde{C}_i(\cdot)\})$ holds. But $\left|V(x, A; \{\widetilde{C}_i(\cdot)\})\right| = 1 < \left|V(y, A; \{\tilde{C}_i(\cdot)\})\right| = 3$ and $x \notin \tilde{C}(A)$ violating locality L. Since F is not local $F \notin \Lambda^M$ and $F \notin \Lambda^{Ne_2}$ by Theorem **??**.

($F \in \Lambda^{NVP}$) Take any $x \in X \in \mathcal{A}$. Let profile $\{C_i(\cdot)\}$ be such that $\forall i \in N \quad |C_i(\cdot)| = 1$ and $|V(x, X; \{C_i(\cdot)\})| = n - 1$ and $\exists y \in X \backslash \{x\}$ such that $|V(y, X; \{C_i(\cdot)\})| = 1$. Then, since $n \geq 3$, $|V(x, X; \{C_i(\cdot)\})| = n - 1 \geq 2 > |V(y, X; \{C_i(\cdot)\})| = 1$, hence $y \notin C(X)$.

($F \in \Lambda^{RA}$) Let N be partitioned into two sets ω_1 and ω_2, i.e. $N = \omega_1 \cup \omega_2$ and $\omega_1 \cap \omega_2 = \emptyset$. Let $x \in C_{\omega_1}(A) \cap C_{\omega_2}(A) \neq \emptyset$. Then since $\forall y \in A \quad |V_{\omega_1}(x, A; \{C_i(\cdot)\})| \geq |V_{\omega_1}(y, A; \{C_i(\cdot)\})|$ and $|V_{\omega_2}(x, A; \{C_i(\cdot)\})| \geq |V_{\omega_2}(y, A; \{C_i(\cdot)\})|$ I have $|V_N(x, A; \{C_i(\cdot)\})| \geq |V_N(y, A; \{C_i(\cdot)\})|$. Therefore, $x \in C_N(A)$.

Let $x \in C_N(A)$. Let $x \notin C_{\omega_1}(A) \cap C_{\omega_2}(A) \neq \emptyset$. Let $z \in C_{\omega_1}(A) \cap C_{\omega_2}(A)$. Then, $|V_{\omega_1}(z, A; \{C_i(\cdot)\})| \geq |V_{\omega_1}(x, A; \{C_i(\cdot)\})|$ and $|V_{\omega_2}(z, A; \{C_i(\cdot)\})| \geq |V_{\omega_2}(x, A; \{C_i(\cdot)\})|$with at least one strict inequality, implying that $x \notin C_N(A)$ which is a contradiction. Hence $x \in C_{\omega_1}(A) \cap C_{\omega_2}(A)$. □

9) *$k-$Majority rule*

Theorem 50 *(Aizerman and Aleskerov, 1986) The Symmetrically Central Class in Λ_{FVR} coincides with the class of k-majority rules*[21].

Theorem 51 *(i) k-majority rule satisifes Positive and Negative Pareto conditions;*
(ii) k-majority satisfies No Veto Power if and only if $1 < k \leq n$;
(iii) k-majority satisfies Reinforcement Axiom if and only if $k = n$.

Proof : ($F \in \Lambda^{U^+}$) Take any $x \in X \in \mathcal{A}$ and any $1 \leq k \leq n$. Let profile $\{C_i(\cdot)\}$be such that $\forall i \in N \quad x \in C_i(X)$. Since $|V(x, X; \{C_i(\cdot)\})| = n \geq k$, $x \in C(X)$.

[21] In the referred paper these rules are investigated in a more general context.

($F \in \Lambda^{U^-}$) Take any $x \in X \in \mathcal{A}$ and any $1 \leq k \leq n$. Let profile $\{C_i(\cdot)\}$ be such that $\forall i \in N \quad x \notin C_i(X)$. Since $|V(x, X; \{C_i(\cdot)\})| = 0 < k$, $x \notin C(X)$.

($k = 1 \Leftrightarrow F \notin \Lambda^{NVP}$) Take any $x \in X \in \mathcal{A}$. Let profile $\{C_i(\cdot)\}$ be such that $\forall i \in N \quad |C_i(\cdot)| = 1$ and $|V(x, X; \{C_i(\cdot)\})| = n - 1$ and $\exists y \in X \backslash \{x\}$ such that $|V(y, X; \{C_i(\cdot)\})| = 1$. Since $|V(y, X; \{C_i(\cdot)\})| = 1 \geq k = 1$, $y \in C(X)$ hence $F \notin \Lambda^{NVP}$. If $1 < k \leq n$ then $|V(y, X; \{C_i(\cdot)\})| = 1 < k$ then $y \notin C(X)$ hence $F \in \Lambda^{NVP}$.

($1 \leq k < n \Rightarrow F \notin \Lambda^{RA}$) First consider the following example where $A = \{x, y, z\}$, $n = 5$ and $k = 2$ which can be generalized to any k and where $\omega_1 = \{1, 2, 3, 4\}$ and $\omega_2 = \{5, 6, 7, 8\}$.

X	2	2	$C_{\omega_1}(\cdot)$
A	$\{x\}$	$\{y\}$	$\{x, y\}$

X	2	2	$C_{\omega_2}(\cdot)$
A	$\{y\}$	$\{y\}$	$\{y\}$

X	2	2	2	2	$C(\cdot)$
A	$\{x\}$	$\{y\}$	$\{y\}$	$\{y\}$	$\{x, y\}$

Now consider the generalized example.

Let $A = \{x, y, z\}$ and $\omega_1 = \{1, 2, ..., k, ..., 2k\}$.

X	k	k	$C_{\omega_1}(\cdot)$
A	$\{x\}$	$\{y\}$	$\{x, y\}$

X	k	k	$C_{\omega_2}(\cdot)$
A	$\{y\}$	$\{y\}$	$\{y\}$

X	k	k	k	k	$C(\cdot)$
A	$\{x\}$	$\{y\}$	$\{y\}$	$\{y\}$	$\{x, y\}$

In this example n is considered to be even but this restriction can be relaxed by just adding one more individiual to ω_1 or ω_2 with an identical choice function according to the coalition to which it is to be added. It can be seen that $C_{\omega_1}(A) = \{x, y\} = B_1$ and $C_{\omega_2}(A) = \{y\} = B_2$. So $B_1 \cap B_2 = \{y\} \neq \emptyset$ but $B_1 \cap B_2 \neq \{x, y\} = C_N(A)$.

($k = n \Rightarrow F \in \Lambda^{RA}$) Let $x \in C_{\omega_1}(A) \cap C_{\omega_2}(A) \neq \emptyset$. Then $\forall i \in N = \omega_1 \cup \omega_2 \quad x \in C_i(A)$. Hence $x \in C_N(A)$. Now, let $x \in C_N(A)$. Then $\forall i \in N = \omega_1 \cup \omega_2 \quad x \in C_i(A)$. Hence $x \in C_{\omega_1}(A)$ and $x \in C_{\omega_2}(A)$. Then $x \in C_{\omega_1}(A) \cap C_{\omega_2}(A)$.

□

10) *Voting with Veto*

Theorem 52 *(Aleskerov, 1999) Voting wtih Veto procedure belongs to the Central Class.*

Theorem 53 *Voting with Veto satisfies Positive and Negative Pareto, No veto power conditions and Reinforcement Axiom. It does not satisfy Anonymity condition.*

Proof : ($F \in \Lambda^{U^+}$) Let $\forall i \in N \quad a \in C_i(A)$ then $\rceil \exists i \in \omega_0$ which is a vetoer and $|V(a, A; \{C_i(\cdot)\})| = n \geq \lceil n/2 \rceil$ hence $a \in C(A)$.

($F \in \Lambda^{U^-}$) Let $\forall i \in N \quad a \notin C_i(A)$ then $\exists i \in \omega_0$ such that $C_i(A) = \emptyset$. Then $a \notin C(A)$ by definition of F.

($F \notin \Lambda^{An}$) Consider the following example where $A = \{x, y, z\}$. Let set of vetoers $\omega_0 = \{1, 2\}$ and $N \backslash \omega_0 = \{3, 4, 5\}$ and therefore $\lceil n/2 \rceil = 3$.

X	$C_1(\cdot)$	$C_2(\cdot)$	$C_3(\cdot)$	$C_4(\cdot)$	$C_5(\cdot)$	$C(A)$
A	$\{x\}$	$\{x, y\}$	$\{x\}$	$\{y\}$	$\{y\}$	$\{x\}$

X	$C_{\mu(5)}(\cdot)$	$C_{\mu(2)}(\cdot)$	$C_{\mu(3)}(\cdot)$	$C_{\mu(1)}(\cdot)$	$C_{\mu(4)}(\cdot)$	$C_{\mu}(A)$
A	$\{y\}$	$\{x, y\}$	$\{x\}$	$\{x\}$	$\{y\}$	$\{y\}$

Let $\mu : N \to N$ be a bijection on N with the values given in the example. Observe that $C(A) \neq C_\mu(A)$.

($F \in \Lambda^{NVP}$) Take any $x \in X \in \mathcal{A}$. Let profile $\{C_i(\cdot)\}$ be such that $\forall i \in N \quad |C_i(\cdot)| = 1$ and $|V(x, X; \{C_i(\cdot)\})| = n - 1$ and $\exists y \in X \backslash \{x\}$ such that $|V(y, X; \{C_i(\cdot)\})| = 1$. Let $y \in C_{i_0}(X)$. If $i_0 \in \omega_0$ then $y \notin C(X)$ since $|\omega_0| < \lceil n/2 \rceil$ and if $i_0 \notin \omega_0$ then again $y \notin C(X)$ since $\forall i \in \omega_0 \quad y \notin C_i(X)$.

($(|\omega_1| + |\omega_0^2|) > \lceil n/2 \rceil \Rightarrow F \notin \Lambda^{RA}$) For this proof it is additionally necessary to partition N into two sets other than the partition of vetoers and others, that is ω_0 and $N \backslash \omega_0$. For this purpose ω_1 and ω_2 will be used. Then, $\omega_0^1 = \omega_1 \cap \omega_0$ (resp. $\omega_0^2 = \omega_2 \cap \omega_0$) will denote vetoers in ω_1 (resp. in ω_2) and $\omega_1 \backslash \omega_0^1$ (resp. $\omega_2 \backslash \omega_0^2$)) will denote others respectively.

First following Lemma is proved.

Lemma 3 $\lceil x \rceil + \lceil y \rceil \geq \lceil x + y \rceil$ *where* $x, y \geq 0$.

Proof : Let x and y be nonnegative real numbers. Consider the fractional portions of x and y, i.e., $x = x_1 + \epsilon_1$ and $y = y_1 + \epsilon_2$ where x_1 and y_1 are nonnegative integers and $0 \leq \epsilon_i < 1$, $i \in \{1, 2\}$. Then there are three cases. (1) Let $\forall i \in \{1, 2\} \quad \epsilon_i \neq 0$. Then $\lceil x \rceil + \lceil y \rceil = \lceil x_1 + \epsilon_1 \rceil + \lceil y_1 + \epsilon_2 \rceil = x_1 + y_1 + 2 \geq \lceil x_1 + y_1 + \epsilon_1 + \epsilon_2 \rceil = \lceil x + y \rceil$ since $0 \leq \epsilon_1 + \epsilon_2 < 2$. (2) Let $\epsilon_2 = 0$. Then $\lceil x \rceil + \lceil y \rceil = \lceil x_1 + \epsilon_1 \rceil + \lceil y_1 \rceil = x_1 + y_1 + 1 \geq \lceil x_1 + y_1 + \epsilon_1 \rceil = \lceil x + y \rceil$ since $0 \leq \epsilon_1 < 1$. (3) Similar proof for $\epsilon_1 = 0$. □

Now, $F \notin \Lambda^{RA}$ is to be proved.

		ω_1		
X	ω_0^1	$\lceil |\omega_1| /2 \rceil - |\omega_0^1|$	$|\omega_1| - \lceil |\omega_1| /2 \rceil$	$C(\cdot)$
$\{x, y, z\}$	$\{x, y\}$	$\{x, y\}$	$\{x\}$	$\{x, y\}$

For subsociety ω_1 above, $\forall i \in \omega_0^1 \quad x, y \in C_i(A)$ and $|V_{\omega_1}(x, A; \{C_i(\cdot)\})| = |\omega_1| \geq \lceil |\omega_1| /2 \rceil$ and$|V_{\omega_1}(y, A; \{C_i(\cdot)\})| = \lceil |\omega_1| /2 \rceil \geq \lceil |\omega_1| /2 \rceil$ hence $C_{\omega_1}(A) = \{x, y\}$.

		ω_2	
X	ω_0^2	$|\omega_2| - |\omega_0^2|$	$C(\cdot)$
$\{x, y, z\}$	$\{x, y\}$	$\{y\}$	$\{y\}$

For subsociety N_2 above, $\forall i \in \omega_0^2 \quad x, y \in C_i(A)$ and $|V_{\omega_2}(x, A; \{C_i(\cdot)\})| = |\omega_0^2| < \lceil |\omega_2| /2 \rceil$ and$|V_{\omega_2}(y, A; \{C_i(\cdot)\})| = |\omega_2| \geq \lceil |\omega_2| /2 \rceil$ hence $C_{\omega_2}(A) = \{y\}$.

As one can see $\{y\} = C_{\omega_1}(A) \cap C_{\omega_2}(A) \neq \emptyset$. Now if join two subsocieties from presentation A the choice functions are as follows:

$$\begin{array}{ccccc} & & N & & \\ \underline{\omega_0} & \underline{\lceil |\omega_1|/2\rceil - |\omega_0^1|} & \underline{\lceil |\omega_2|/2\rceil - |\omega_0^2|} & \underline{|\omega_1| - \lceil |\omega_1|/2\rceil} & \underline{C(\cdot)} \\ \{x,y\} & \{x,y\} & \{y\} & \{x\} & \{x,y\} \end{array}$$

As one can see, under the assumption $|\omega_1| + |\omega_0^2| \geq \lceil n/2\rceil$, $x \in C_N(A)$ hence $C_N(A) \neq C_{\omega_1}(A) \cap C_{\omega_2}(A)$. Note that, preserving the main idea, this assumption could be modified to $|V_{\omega_1}(x, A; \{C_i(\cdot)\})| + |\omega_0^2| \geq \lceil n/2\rceil$ under the restriction $V_{\omega_1}(x, A; \{C_i(\cdot)\}) \supset \omega_0^1$ and $V_{\omega_2}(x, A; \{C_i(\cdot)\}) \supseteq \omega_0^2$. $\square$

5.2.2 Rationality Constraints for FVRs

8) *Approval Voting*

Theorem 54 *Approval Voting satisfies Non-emptiness. It does not satisfy Weak-non resoluteness, Inverse and Direct Condorcet, Independence of Outcast, Fixed Point conditions and Weak Axiom of Revealed Preferences. When the domain of definition for individual opinions satisfy Arrow 's Choice Axiom in strong version,it does not satisfy Dual Heritage constraint.*

Proof : ($F \in \Lambda(NE)$) Since $\forall i \in N \quad C_i(\cdot) \in ACA^+$, $\exists x \in A$ such that $|V(x, A; \{C_i(A)\})| \geq 1$. Since $\max\limits_{x\in A} |V(x, A; \{C_i(A)\})|$ exists, $C(A) \neq \emptyset$.

($F \notin \Lambda(ACA^+ \cap WNR, WNR)$) Consider the following example.

X	$C_1(\cdot)$	$C_2(\cdot)$	$C_3(\cdot)$	$C(\cdot)$
$\{x,y,z\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\{x,y,z\}$

In this example $\forall i \in N \quad C_i(A) \subset A$ but $C(A) = A \not\subset A$.

($F \notin \Lambda(Con^-)$) Consider the following example.

X	$C_1(\cdot)$	$C_2(\cdot)$	$C_3(\cdot)$	$C_4(\cdot)$	$C(\cdot)$
$\{x,y,z\}$	$\{x,y\}$	$\{z\}$	$\{x\}$	$\{z\}$	$\{x,z\}$
$\{x,y\}$	$\{x,y\}$	$\{y\}$	$\{x\}$	$\{y\}$	$\{y\}$
$\{x,z\}$	$\{x\}$	$\{z\}$	$\{x\}$	$\{z\}$	$\{x,z\}$
$\{y,z\}$	$\{y\}$	$\{z\}$	$\{y\}$	$\{z\}$	$\{y,z\}$

In this example, $x \in C(\{x,y,z\})$ but $x \notin C(\{x,y\})$. Since $F \notin \Lambda(Con^-)$, then $F \notin \Lambda(H)$ and $F \notin \Lambda(ACA)$ by Theorem **??**.

($F \notin \Lambda(Con^+)$) Consider the previous example. In this example, $C(\{x,y\}) \cap C(\{z,y\}) = \{y\} \not\subseteq C(\{x,y,z\}) = \{x,z\}$. Since $F \notin \Lambda(Con^+)$, then $F \notin \Lambda(C)$ by Theorem **??**.

($F \notin \Lambda(O)$) Consider the following example.

X	$C_1(\cdot)$	$C_2(\cdot)$	$C_3(\cdot)$	$C(\cdot)$
$\{x,y,z\}$	$\{x,y\}$	$\{x\}$	$\{z\}$	$\{x\}$
$\{x,y\}$	$\{x,y\}$	$\{x\}$	$\{y\}$	$\{x,y\}$
$\{x,z\}$	$\{x\}$	$\{x\}$	$\{z\}$	$\{x\}$
$\{y,z\}$	$\{y\}$	$\{y\}$	$\{z\}$	$\{y\}$

In this example, $C(\{x,y,z\}) = \{x\} \neq C(\{x,y\}) = \{x,y\}$. Since $F \notin \Lambda(O)$, then, $F \notin \Lambda(PI)$ by Theorem **??**.

($F \notin \Lambda(ACA^*, H^-)$) Consider the following example for odd n. Notice that since $\forall i \in N \quad C_i(\cdot) \in ACA^*$, empty choice is allowed.

X	$\lceil n/2 \rceil - 1$	$\lceil n/2 \rceil - 1$	1	$C(\cdot)$
$\{x,y,z\}$	$\{x,y\}$	$\{z\}$	$\{y\}$	$\{y\}$
$\{x,y\}$	$\{x,y\}$	$\emptyset$	$\{y\}$	$\{y\}$
$\{x,z\}$	$\{x\}$	$\{z\}$	$\emptyset$	$\{x,z\}$
$\{y,z\}$	$\{y\}$	$\{z\}$	$\{y\}$	$\{y\}$

In this example, $\{x,z\} \subseteq A$ but $C(\{x,z\}) \nsubseteq C(A)$.

Consider the following example for even n.

X	$n/2$	$n/2$	$C(\cdot)$
$\{x,y,z\}$	$\{x,y\}$	$\{y,z\}$	$\{y\}$
$\{x,y\}$	$\{x,y\}$	$\{y\}$	$\{y\}$
$\{x,z\}$	$\{x\}$	$\{z\}$	$\{x,z\}$
$\{y,z\}$	$\{y\}$	$\{y,z\}$	$\{y\}$

In this example, $\{x,z\} \subseteq A$ but $C(\{x,z\}) \nsubseteq C(A)$.

Since $F \notin \Lambda(ACA^*, H^-)$, then $F \notin \Lambda(ACA^*, ACA^*)$ by Theorem **??**.

($F \notin \Lambda(FP)$) Consider the following example.

X	$C_1(\cdot)$	$C_2(\cdot)$	$C_3(\cdot)$	$C(\cdot)$
$\{x,y,z\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\{x,y,z\}$
$\{x,y\}$	$\{x\}$	$\{y\}$	$\{y\}$	$\{y\}$
$\{x,z\}$	$\{x\}$	$\{x\}$	$\{z\}$	$\{x\}$
$\{y,z\}$	$\{z\}$	$\{y\}$	$\{z\}$	$\{z\}$

In this example, since $\forall i \in N \quad C_i(\cdot) \in ACA^+ \cap FP$ but $C(\cdot) \notin FP$ since $\rceil \exists a \in A$ such that $\forall X \subseteq A \quad a \in X \Rightarrow a \in C(X)$.

($n \geq 4 \Rightarrow F \notin \Lambda(WARP)$) Consider the following example where $n \geq 4$.

X	1	$n-2$	1	$C(\cdot)$
$\{x,y,z\}$	$\{x,y\}$	$\{z\}$	$\{x\}$	$\{x,z\}$
$\{x,y\}$	$\{x,y\}$	$\{y\}$	$\{x\}$	$\{y\}$
$\{x,z\}$	$\{x\}$	$\{z\}$	$\{x\}$	$\{x,z\}$
$\{y,z\}$	$\{y\}$	$\{z\}$	$\{y\}$	$\{y,z\}$

In this example, since $C(A) = \{x, z\}$, xGy and zGy. But, since $C(\{x, y\}) = \{y\}$, I also have yGx which contradicts acyclicity of G. $\Box$

9) *k-Majority rule*

Theorem 55 *(Aleskerov, forthcoming) k-majority rules are,*
(i) *Non-empty rules if and only if* $|A| < \lceil n/(k-1) \rceil$ *,*
(ii) *Weak non-resolute rules if and only if*
$|A| < \lceil n/(n-k) \rceil$ *,*
where a rule F *is non-empty (resp. weak non-resolute) if and only if* $F \in \Lambda(NE, NE)$ *(resp.* $F \in \Lambda(WNR, WNR)$*).*

Note that in our notation $\Lambda(NE) = \Lambda(ACA^+, NE)$ and I check for $\Lambda(ACA^+ \cap WNR, WNR)$.

Theorem 56 (i) *k-majority rule satisfies Non-emptiness (or Fixed point condition) if and only if* $|A| < \lceil n/(k-1) \rceil$ *and if* $|A| < \lceil n/(n-k) \rceil$ *then k-majority rule satisfies Weak non-resoluteness;*
(ii) *k-majority rule satisfies Heritage Condition;*
(iii) *When the domain of definition is* ACA^**, k-majority rule satisfies* ACA^**, i.e., Arrow 's choice Axiom in Strong version;*
(iv) *For* $1 \leq k < n$*, k-majority rule does not satisfy Direct Condorcet condition, for* $k = n$*, k-majority rule satisfies Concordance condition;*
(v) *k-majority rule satisfies Independence of Outcast if and only if* $k = 1$*;*
(v) *k-majority rule satisfies Weak Axiom of Revealed Preference if and only if* $k = n$*.*

Proof : ($|A| < \lceil n/(k-1) \rceil \Rightarrow F \in \Lambda(NE)$) Let $|A| < \lceil n/(k-1) \rceil$ then $F \in \Lambda(NE, NE)$ by Theorem **??**. Then $\forall i \in N \quad C_i(\cdot) \in NE \Rightarrow C(\cdot) \in NE$. Since $ACA \cap NE = ACA^+ \subset NE$, $\forall i \in N \quad C_i(\cdot) \in ACA^+ \Rightarrow C(\cdot) \in NE$. Hence $F \in \Lambda(ACA^+, NE)$.
($|A| < \lceil n/(n-k) \rceil \Rightarrow F \in \Lambda(ACA^+ \cap WNR, WNR)$) Let $|A| < \lceil n/(n-k) \rceil$ then $F \in \Lambda(WNR, WNR)$ by Theorem **??**. Then $\forall i \in N \quad C_i(\cdot) \in WNR \Rightarrow C(\cdot) \in WNR$. Since $ACA^+ \cap WNR \subset WNR$, $\forall i \in N \quad C_i(\cdot) \in ACA^+ \cap WNR \Rightarrow C(\cdot) \in WNR$. Hence $F \in \Lambda(ACA^+ \cap WNR, WNR)$.
($F \in \Lambda(H)$) Let $X' \subseteq X$. Take any $x \in X' \subseteq X$ such that $x \in C(X)$. Then $|V(x, X, \{C_i(\cdot)\})| \geq k$. Since $\forall i \in N \quad C_i(\cdot) \in ACA^+ \subset H$, $x \in C_i(X) \Rightarrow x \in C_i(X) \cap X' \neq \emptyset \Rightarrow x \in C_i(X')$. Then, $k \leq |V(x, X; \{C_i(\cdot)\})| \leq |V(x, X'; \{C_i(\cdot)\})|$, $x \in C(X')$. Since $F \in \Lambda(H)$, then $F \in \Lambda(Con^-)$ by Theorem **??**.
Now the following Lemma is introduced.

Lemma 4 $F \in \Lambda(ACA^*, H)$.

Proof : Let $X' \subseteq X$. Take any $x \in X' \subseteq X$ such that $x \in C(X)$. Then $|V(x, X, \{C_i(\cdot)\})| \geq k$. Since $\forall i \in N \quad C_i(\cdot) \in ACA^* \subset H$, $x \in C_i(X) = X \cap A^*$

$\Rightarrow x \in X' \cap A^* = C_i'(X)$. Then, $k \leq |V(x, X; \{C_i(\cdot)\})| \leq |V(x, X'; \{C_i(\cdot)\})|$, hence $x \in C(X')$. □

($F \in \Lambda(ACA^*, H^-)$) Let $X' \subseteq X$. Then let $x \in C(X')$ implying that $|V(x, X'; \{C_i(\cdot)\})| \geq k$. Since $\forall i \in N \quad C_i(\cdot) \in ACA^* \subset H^-$, $\forall i \in N \quad C_i(X') \subseteq C_i(X)$. Then
$|V(x, X; \{C_i(\cdot)\})| \geq |V(x, X'; \{C_i(\cdot)\})| \geq k$ implying that $x \in C(X)$. Since $F \in \Lambda(ACA^*, H^-) \cap \Lambda(ACA^*, H)$, then $F \in \Lambda(ACA^*, ACA^*)$ by Theorem **??**.

($F \notin \Lambda(Con^+)$) For the case of $k = 1$ consider the following example.

X	1	$n-1$	$C(\cdot)$
$\{x,y,z\}$	$\{z\}$	$\{y\}$	$\{y,z\}$
$\{x,y\}$	$\{x\}$	$\{y\}$	$\{x,y\}$
$\{x,z\}$	$\{z\}$	$\{x\}$	$\{x,z\}$
$\{y,z\}$	$\{z\}$	$\{y\}$	$\{y,z\}$

In this example, $x \in C(\{x,x\}) \cap C(\{y,x\}) \cap C(\{z,x\})$ but $x \notin C(\{x,y,z\})$.

For the case of $(1 < k < n-1)$ consider the following example[22] where $n = 4\lceil k/2 \rceil$.

X	$\lceil k/2 \rceil$	$\lceil k/2 \rceil$	$\lceil k/2 \rceil$	$\lceil k/2 \rceil$	$C(\cdot)$
$\{x,y,z\}$	$\{x,y\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\{x,y\}$
$\{x,y\}$	$\{x,y\}$	$\{x\}$	$\{y\}$	$\cdot$	$\{x,y\}$
$\{x,z\}$	$\{x\}$	$\{x\}$	$\{z\}$	$\{z\}$	$\{x,z\}$
$\{y,z\}$	$\{y\}$	$\{z\}$	$\{y\}$	$\{z\}$	$\{y,z\}$

In this example, $z \in C(\{x,z\}) \cap C(\{y,z\}) \cap C(\{z,z\})$ but $z \notin C(\{x,y,z\})$. For the case of $1 \leq k < n$ since $F \notin \Lambda(Con^+)$, then $F \notin \Lambda(ACA)$ and $F \notin \Lambda(C)$ by Theorem **??**.

For the case of $(k = n-1)$ consider the following example where $n = 3$.

X	1	1	1	$C(\cdot)$
$\{x,y,z\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\emptyset$
$\{x,y\}$	$\{x\}$	$\{y\}$	$\{x\}$	$\{x\}$
$\{x,z\}$	$\{x\}$	$\{x\}$	$\{z\}$	$\{x\}$
$\{y,z\}$	$\{y\}$	$\{y\}$	$\{z\}$	$\{y\}$

In this example, $x \in C(\{x,x\}) \cap C(\{x,y\}) \cap C(\{x,z\})$ but $x \notin C(\{x,y,z\})$.

($k = n \Rightarrow F \in \Lambda(C)$) Let $X' \subseteq X$. Let $x \in C(X_1) \cap C(X_2)$. Then $\forall i \in N \quad x \in C(X_1) \cap C(X_2) \neq \emptyset$, then since $\forall i \in N \quad C_i(\cdot) \in ACA^+$, $\forall i \in N \quad x \in C_i(X_1 \cup X_2) = C_i(X_1) \cap C_i(X_2)$, then $x \in C(X_1 \cup X_2)$. Since $F \in \Lambda(C)$, then $F \in \Lambda(Con^+)$ by Theorem **??**.

($k = 1 \Rightarrow F \in \Lambda(O)$) Assume $C(X) \subseteq X' \subseteq X$. Since when $k = 1$ $F \in \Lambda(NE)$, $\forall X \in \mathcal{A} \quad C(X) \neq \emptyset$. Let $x \in C(X)$. Then $x \in X'$ and therefore $x \in C(X) \cap X'$. By $F \in \Lambda(H)$, then $x \in C(X')$. Now, let $x \in C(X')$. Then $\exists i \in N$ such that $x \in C_i(X')$. By $\forall i \in N \quad C_i(X) \in ACA^+$, $C_i(X) \neq \emptyset$. Let $C_i(X) \cap X' = \emptyset$. Then $y \in C_i(X)$, and

[22] By Lemma **??** $\lceil k/2 \rceil + \lceil k/2 \rceil \geq k$.

then $y \in C(X)$. But $y \notin C(X')$ which contradicts $C(X) \subseteq X'$. Then $C_i(X) \cap X' \neq \emptyset$. Then $x \in C_i(X') = C_i(X) \cap X'$ by ACA. Then $x \in C_i(X)$, and then $x \in C(X)$ satisfying O. Since $F \in \Lambda(O) \cap \Lambda(H)$, then $F \in \Lambda(PI)$ by Theorem **??**.

($1 < k \leq n \Rightarrow F \notin \Lambda(O)$) First consider the following example for teh case of $k = 2$.

X	1	1	1	$C(\cdot)$
$\{x, y, z\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\emptyset$
$\{x, y\}$	$\{x\}$	$\{y\}$	$\{y\}$	$\{y\}$
$\{x, z\}$	$\{x\}$	$\{z\}$	$\{z\}$	$\{z\}$
$\{y, z\}$	$\{y, z\}$	$\{y\}$	$\{z\}$	$\{y, z\}$

In this example, $C(\{x, y, z\}) = \emptyset \subset \{y, z\} \subset \{x, y, z\}$ but $C(\{y, z\}) = \{y, z\} \neq \emptyset$.

Now consider the following example for $2 < k < n$ where $n = 2k - 2$.

X	$k-2$	$k-2$	1	1	$C(\cdot)$
$\{x, y, z\}$	$\{x, y\}$	$\{z\}$	$\{x\}$	$\{z\}$	$\emptyset$
$\{x, y\}$	$\{x, y\}$	$\{y\}$	$\{x\}$	$\{y\}$	$\cdot$
$\{x, z\}$	$\{x\}$	$\{z\}$	$\{x\}$	$\{z\}$	$\emptyset$
$\{y, z\}$	$\{y\}$	$\{z\}$	$\{z\}$	$\{z\}$	$\{z\}$

In this example, $C(\{x, y, z\}) = \emptyset \subseteq \{y, z\} \subseteq \{x, y, z\}$ but $C(\{y, z\}) = \{z\} \neq \emptyset$. Now for $k = n$ consider the following example.

X	1	$n-1$	$C(\cdot)$
$\{x, y, z\}$	$\{y\}$	$\{z\}$	$\emptyset$
$\{x, y\}$	$\{y\}$	$\{y\}$	$\{y\}$

In this example, $C(A) \subset \{x, y\} \subset A$ but $\emptyset = C(A) \neq C(\{x, y\}) = \{y\}$. Since $F \notin \Lambda(O)$, then $F \notin \Lambda(ACA)$ and $F \notin \Lambda(PI)$ by Theorem **??**.

($1 \leq k < n-1 \Rightarrow F \notin \Lambda(WARP)$) Consider the following example where $n-k \geq k$.

X	k	$n-k$	$C(\cdot)$
$\{x, y, z, w\}$	$\{x\}$	$\{y\}$	$\{x, y\}$
$\{x, y, z\}$	$\{x\}$	$\{y\}$	$\{x, y\}$
$\{x, y, w\}$	$\{x\}$	$\{y\}$	$\{x, y\}$
$\{x, z, w\}$	$\{x\}$	$\{z\}$	$\{x, z\}$
$\{y, z, w\}$	$\{w\}$	$\{y\}$	$\{w, y\}$

Since $z \in C(\{x, z, w\}) = \{x, z\}$ and $w \notin C(\{x, z, w\})$, zGw. But then $w \in C(\{y, z, w\}) = \{w, y\}$ and $z \notin C(\{y, z, w\})$, wGz violating acyclicity of G.

($k = n-1 \Rightarrow F \notin \Lambda(WARP)$) Consider the following example where $n = 3$ and $k = n-1 = 2$.

$\underline{X}$	$\underline{1}$	$\underline{1}$	$\underline{1}$	$\underline{C(\cdot)}$
$\{x,y,z,w,t\}$	$\{x\}$	$\{y\}$	$\{w\}$	$\emptyset$
$\{x,y,z,w\}$	$\{x\}$	$\{y\}$	$\{w\}$	$\emptyset$
$\{x,y,z,t\}$	$\{x\}$	$\{y\}$	$\{x\}$	$\{x\}$
$\{x,z,w,t\}$	$\{x\}$	$\{w\}$	$\{w\}$	$\{w\}$
$\{x,y,w,t\}$	$\{x\}$	$\{y\}$	$\{w\}$	$\emptyset$
$\{y,z,w,t\}$	$\{y\}$	$\{y\}$	$\{w\}$	$\{y\}$

In this example, $xGyGwGx$.

($k = n \Rightarrow F \in \Lambda(WARP)$) Construct P_i corresponding to $C_i(\cdot)$ for each voter in N such that $C_i(X) = \{x \in X : \rceil \exists y \in X \text{ such that } yP_ix\}$. Since $\forall i \in N \quad C_i(X) \in ACA^+$ and by Theorem **??**, these binary relations are weak orders, i.e., $\forall i \in N \quad P_i \in \mathcal{W}$.

Now, suppose that $\exists x_1, x_2, ..., x_m$ such that

$$x_1 \in C(X_1) \text{ and } x_2 \in X_1 \backslash C(X_1)$$
$$x_2 \in C(X_2) \text{ and } x_3 \in X_2 \backslash C(X_2)$$
$$\cdot$$
$$\cdot$$
$$x_m \in C(X_m) \text{ and } x_1 \in X_m \backslash C(X_m).$$

i.e., $\exists x_1Gx_2G, ..., Gx_mGx_1$ where $\forall x, y \in A \quad xGy \Leftrightarrow [x \in C(X) \text{ and } y \in X \backslash C(X)]$ for some $X \in \mathcal{A}$.

Since x_1Gx_2, then $\exists X_1 \in \mathcal{A}$ such that $x_1 \in C(X_1)$ and $x_2 \notin C(X_1)$ where $x_1, x_2 \in X_1$. Then $\forall i \in N \quad x_1 \in C_i(X_1)$ and $\exists i_1 \in N \quad$ such that $x_2 \notin C_{i_1}(X_1)$. Since P_{i_1} is a weak order, either $x_1P_{i_1}x_2$ or $[x_1\bar{P}_{i_1}x_2$ and $x_2\bar{P}_{i_1}x_1$ and $zP_{i_1}x_2]$ for some $z \in X_1$. Suppose $x_1\bar{P}_{i_1}x_2$. But this implies $zP_{i_1}x_1$ since P_{i_1} is a weak order. Since $zP_{i_1}x_1$ is impossible by the facts that $z \in X_1$ and $\forall i \in N \quad x_1 \in C_i(X_1)$, there is a contradiction, so $x_1P_{i_1}x_2$. Since x_2Gx_3, then by similar arguments $\forall i \in N \quad x_2 \in C_i(X_2)$ and $\exists i_2 \in N \quad$ such that $x_3 \notin C_{i_2}(X_2)$. Since P_{i_2} is a weak order, $x_2P_{i_2}x_3$. Similarly, $x_3P_{i_3}x_4, x_4P_{i_4}x_5$, and then $x_mP_{i_m}x_1$.

Now consider P_{i_1} which is a weak order. It was stated that $x_1P_{i_1}x_2$. Consider now x_3. Since P_{i_1} is a weak order, either $x_1P_{i_1}x_3$ or $x_3P_{i_1}x_2$. Since $x_3 \in X_2$ and $x_2 \in C(X_2) = \bigcap_{i \in N} C_i(X_2)$, $x_3\bar{P}_{i_1}x_2$ so it must be that $x_1P_{i_1}x_3$. Consider then x_4. Since P_{i_1} is a weak order, either $x_1P_{i_1}x_4$ or $x_4P_{i_1}x_3$. Since $x_4 \in X_3$ and $x_3 \in C(X_3) = \bigcap_{i \in N} C_i(X_3)$, $x_4\bar{P}_{i_1}x_3$ so it must be that $x_1P_{i_1}x_4$. Consider, similarly all alternatives up to x_{m-1}. Similarly $x_1P_{i_1}x_{m-1}$ must hold.

This time consider x_m. Since P_{i_1} is a weak order, either $x_1P_{i_1}x_m$ or $x_mP_{i_1}x_{m-1}$. Since $x_m \in X_{m-1}$ and $x_{m-1} \in C(X_{m-1}) = \bigcap_{i \in N} C_i(X_{m-1})$, $x_m\bar{P}_{i_1}x_{m-1}$ so it must be that $x_1P_{i_1}x_m$. But $x_1 \in X_m \backslash C(X_m)$ and $x_m \in C(X_m) = \bigcap_{i \in N} C_i(X_m)$, $x_1\bar{P}_{i_1}x_m$ which leads to a contradiction. So G is acyclic. □

10) *Voting with Veto*

Theorem 57 *Voting with Veto satisfies Weak non-resoluteness and Heritage. It does not satisfy Non-emptiness, Independence of Outcast, Direct Condorcet and Weak Axiom of Revealed Preferences. It satisfies Arrow 's choice Axiom in strong version when the domain of definition also satisfiesthis axiom.*

Proof : ($F \notin \Lambda(NE)$) Consider the example below. In this example, $\forall i \in N \quad C_i(\cdot) \in ACA^+$, but $C(A) = \emptyset$. Since $F \notin \Lambda(NE)$, $F \notin \Lambda(FP)$ by Theorem **??**.

($F \in \Lambda(ACA^+ \cap WNR, WNR)$) Since $\forall i \in N \quad C_i(\cdot) \in ACA^+ \cap WNR$, $\exists x \in A$ such that $x \notin C_i(A)$ for some $i \in \omega_0$ hence $x \notin C(A) \subset A$.

($F \notin \Lambda(Con^+)$) Consider the following example where $N = \{1,2,3,4,5\}$ and $\omega_0 = \{1,2\}$.

X	$C_1(\cdot)$	$C_2(\cdot)$	$C_3(\cdot)$	$C_4(\cdot)$	$C_5(\cdot)$	$C(\cdot)$
$\{x,y,z,w\}$	$\{x\}$	$\{x,w\}$	$\{y\}$	$\{z\}$	$\{y\}$	$\emptyset$
$\{x,y,z\}$	$\{x\}$	$\{x\}$	$\{y\}$	$\{z\}$	$\{y\}$	$\emptyset$
$\{x,y,w\}$	$\{x\}$	$\{x,w\}$	$\{y\}$	$\{x\}$	$\{y\}$	$\{x\}$
$\{x,z,w\}$	$\{x\}$	$\{x,w\}$	$\{x\}$	$\{z\}$	$\{z\}$	$\{x\}$
$\{y,z,w\}$	$\{w\}$	$\{w\}$	$\{y\}$	$\{z\}$	$\{y\}$	$\emptyset$
$\{x,y\}$	$\{x\}$	$\{x\}$	$\{y\}$	$\{x\}$	$\{y\}$	$\{x\}$
$\{x,z\}$	$\{x\}$	$\{x\}$	$\{x\}$	$\{z\}$	$\{z\}$	$\{x\}$
$\{x,w\}$	$\{x\}$	$\{x,w\}$	$\{x\}$	$\{x\}$	$\{w\}$	$\{x\}$
$\{y,z\}$	$\{y\}$	$\{y\}$	$\{y\}$	$\{z\}$	$\{y\}$	$\{y\}$
$\{y,w\}$	$\{w\}$	$\{w\}$	$\{y\}$	$\{w\}$	$\{y\}$	$\{w\}$
$\{z,w\}$	$\{w\}$	$\{w\}$	$\{w\}$	$\{z\}$	$\{z\}$	$\{w\}$

In this example, $x \in C(\{x,y\}) \cap C(\{x,z\}) \cap C(\{x,w\})$ but $x \notin C(\{x,y,z,w\})$. Since $F \notin \Lambda(Con^+)$ then $F \notin \Lambda(C)$ by Theorem **??**.

($F \notin \Lambda(O)$)Consider the previous example. In this example, $C(\{x,y,z,w\}) = \emptyset \subset \{x,y,w\} \subset \{x,y,z,w\}$ but $\emptyset = C(\{x,y,z,w\}) \neq C(\{x,y,w\}) = \{x\}$. Since $F \notin \Lambda(O)$, then $F \notin \Lambda(ACA)$ and $F \notin \Lambda(PI)$ by Theorem **??**.

($F \in \Lambda(H)$) Take any $X' \subseteq X \in \mathcal{A}$. Let $x \in C(X) \cap X'$. Then $\forall i \in \omega_0 \cup \omega \quad x \in C_i(X)$ where $\omega \subseteq N \backslash N_1$ and $|\omega_0| + |\omega| \geq \lceil n/2 \rceil$. Since $\forall i \in N \quad C_i(\cdot) \in ACA^+$, $\forall i \in \omega_0 \cup \omega \quad x \in C_i(X')$ and hence $x \in C(X')$. Since $F \in \Lambda(H)$, then $F \in \Lambda(Con^-)$ by Theorem **??**.

Lemma 5 $F \in \Lambda(ACA^*, H)$.

Proof : Take any $X' \subseteq X \in \mathcal{A}$. Let $x \in C(X) \cap X'$. Then $\forall i \in \omega_0 \cup \omega \quad x \in C_i(X)$ where $\omega \subseteq N \backslash \omega_1$ and $|\omega_0| + |\omega| \geq \lceil n/2 \rceil$. Since $\forall i \in N \quad C_i(\cdot) \in ACA^*$, $\forall i \in \omega_0 \cup \omega \quad x \in C_i(X')$ and hence $x \in C(X')$. $\square$

($F \in \Lambda(ACA^*, H^-)$) Let $X' \subseteq X \in \mathcal{A}$. Then let $x \in C(X')$ implying that $\forall i \in \omega_1 \cup \omega \quad x \in C_i(X')$ where $\{\omega \subseteq N \backslash \omega_1 : |\omega_1| + |\omega| \geq \lceil n/2 \rceil\}$. Since $\forall i \in N \quad C_i(\cdot) \in H^-$, $\forall i \in \omega_1 \cup \omega \quad x \in C_i(X)$ and hence $x \in C(X')$. Since $F \in \Lambda(ACA^*, H^-)$, and $F \in \Lambda(ACA^*, H)$ by Lemma **??**, then $F \in \Lambda(ACA^*, ACA^*)$ by Theorem **??**.

($F \notin \Lambda(WARP)$) Construct P_i corresponding to $C_i(\cdot)$ for each voter in N such that $C_i(X) = \{x \in X :\rceil \exists y \in X$ such that $yP_ix\}$. Since $\forall i \in N \quad C_i(X) \in ACA$, these binary relations are weak orders by Theorem **??**, i.e., $\forall i \in N \quad P_i \in \mathcal{W}$.

Now, suppose that $\exists x_1, x_2, ..., x_m$ such that

$$\begin{array}{c} x_1 \in C(X_1) \text{ and } x_2 \in X_1 \backslash C(X_1) \\ x_2 \in C(X_2) \text{ and } x_3 \in X_2 \backslash C(X_2) \\ . \\ . \\ x_m \in C(X_m) \text{ and } x_1 \in X_m \backslash C(X_m). \end{array}$$

i.e., $\exists x_1 G x_2 G, ..., G x_m G x_1$ where $\forall x, y \in A \quad xGy \Leftrightarrow [x \in C(X) \text{ and } y \in X \backslash C(X)]$ for some $X \in \mathcal{A}$. First following two lemmas are proved where $M = \{1, ..., m\}$.

Lemma 6 $\forall i \in \omega_0 \quad \forall j \in M \quad x_j \in C_i(X_j)$ *and* $x_{j+1} \in C_i(X_j)$ *where* $x_{m+1} = x_1$.

Proof : Since $x_1 G x_2$, then $\exists X_1 \in \mathcal{A}$ such that $x_1 \in C(X_1)$ and $x_2 \notin C(X_1)$. Then $\forall i \in \omega_0 \quad x_1 \in C_i(X_1)$ by definition of the rule and $\exists i_1 \in N \quad$ such that $x_1 \in C_{i_1}(X_1)$ and $x_2 \notin C_{i_1}(X_1)$ for otherwise $x_2 \in C(X_1)$ by Monotonicity of F by Theorem **??**. Suppose $i_1 \in \omega_0$. Since P_{i_1} is a weak order, either $x_1 P_{i_1} x_2$ or $[x_1 \bar{P}_{i_1} x_2$ and $x_2 \bar{P}_{i_1} x_1$ and $z P_{i_1} x_2]$ for some $z \in X_1$. Suppose $x_1 \bar{P}_{i_1} x_2$ and $x_2 \bar{P}_{i_1} x_1$ and $z P_{i_1} x_2$. But this implies $z P_{i_1} x_1$ since P_{i_1} is a weak order. Since $z P_{i_1} x_1$ is impossible by the fact that $\forall i \in \omega_0 \quad x_1 \in C_i(X_1)$, there is a contradiction, so $x_1 P_{i_1} x_2$. Consider now x_3. Since P_{i_1} is a weak order, either $x_1 P_{i_1} x_3$ or $x_3 P_{i_1} x_2$. Since $x_3 \in X_2$ and $x_2 \in C(X_2) \subseteq \bigcap_{i \in \omega_0} C_i(X_2)$, $x_3 \bar{P}_{i_1} x_2$ so it must be that $x_1 P_{i_1} x_3$. Consider then x_4. Since P_{i_1} is a weak order, either $x_1 P_{i_1} x_4$ or $x_4 P_{i_1} x_3$. Since $x_4 \in X_3$ and $x_3 \in C(X_3) \subseteq \bigcap_{i \in \omega_0} C_i(X_3)$, $x_4 \bar{P}_{i_1} x_3$ so it must be that $x_1 P_{i_1} x_4$. Consider, similarly all alternatives up to x_{m-1}. Similarly $x_1 P_{i_1} x_{m-1}$ must hold. This time consider x_m. Since P_{i_1} is a weak order, either $x_1 P_{i_1} x_m$ or $x_m P_{i_1} x_{m-1}$. Since $x_m \in X_{m-1}$ and $x_{m-1} \in C(X_{m-1}) \subseteq \bigcap_{i \in \omega_0} C_i(X_{m-1})$, $x_m \bar{P}_{i_1} x_{m-1}$ so it must be that $x_1 P_{i_1} x_m$. But $x_1 \in X_m \backslash C(X_m)$ and $x_m \in C(X_m) \subseteq \bigcap_{i \in \omega_0} C_i(X_m)$, $x_1 \bar{P}_{i_1} x_m$ which leads to a contradiction. So $i_1 \notin \omega_0$. Hence $\forall i \in \omega_0 \quad x_2 \in C_i(X_1)$.

Since $x_2 G x_3$, then by similar arguments $\forall i \in \omega_0 \quad x_2 \in C_i(X_2)$ and $x_3 \in C_i(X_2)$ and $\exists i_2 \in N \backslash \omega_0 \quad$ such that $x_3 \notin C_{i_2}(X_2)$. Similarly, $\forall i \in \omega_0 \quad x_3 \in C_i(X_3), ...,$ and $\forall i \in \omega_0 \quad x_m \in C_i(X_m)$ and $x_1 \in C_i(X_m)$.

Lemma 7 $\forall i \in \omega_0 \quad \bigcup_{j \in M} C_i(X_j) = C_i(X)$ *where* $X = \bigcup_{j \in M} X_j$.

Proof : By Lemma **??** $\forall i \in \omega_0 \quad \forall j \in M \quad C_i(X_j) \neq \emptyset$. Then $C_i(X) = \emptyset$ is impossible since $\forall j \in M \quad X \supseteq X_j$ and $\forall i \in N \quad C_i(\cdot) \in ACA$.

By Lemma **??** $\forall i \in \omega_0 \quad \forall j \in M \quad x_j \in C_i(X_j)$ and $x_{j+1} \in C_i(X_j)$. Let $C_i(X) \cap X_j = \emptyset$ for some j. Suppose $C_i(X) \cap X_{j+1} \neq \emptyset$. Then $C_i(X) \cap X_{j+1} = C_i(X_{j+1}) \supseteq \{x_{j+1}\}$ and hence $x_{j+1} \in C_i(X) \cap X_j \neq \emptyset$ which is a contradiction. So $C_i(X) \cap X_{j+1} = \emptyset$. Similarly,

$C_i(X) \cap X_{j+2} = \emptyset, ..., C_i(X) \cap X_{m+1} = C_i(X) \cap X_1 = \emptyset$, and $C_i(X) \cap X_{j-1} = \emptyset$. So $\forall j \in M \quad C_i(X) \cap X_j = \emptyset$. But $\emptyset \neq C_i(X) \subseteq X = X_1 \cup ... \cup X_m$ and hence $x \in C_i(X) \Rightarrow x \in X_j \Rightarrow x \in C_i(X_j)$ for some j by ACA which leads to a contradiction. Hence $\forall i \in \omega_0 \quad \forall j \in M \quad C_i(X) \cap X_j \neq \emptyset$ which implies $\forall i \in \omega_0 \quad \forall j \in M \quad C_i(X) \cap X_j = C_i(X_j)$ by $\forall i \in \omega_0 \quad C_i(\cdot) \in ACA$.

Let $x \in \bigcup_{j \in M} C_i(X_j)$ then $x \in C_i(X_j)$ for some j which implies $x \in C_i(X) \cap X_j$. Now let $x \in C_i(X) \subseteq X = X_1 \cup ... \cup X_m$. Then $x \in X_j$ for some j which implies $x \in C_i(X) \cap X_j = C_i(X_j) \subseteq \bigcup_{j \in M} C_i(X_j)$. Hence $\bigcup_{j \in M} C_i(X_j) = C_i(X)$. □

By Lemmas **??** and **??** and by $\forall i \in \omega_0 \quad C_i(\cdot) \in ACA$, $\forall i \in \omega_0 \quad \forall j \in M \quad x_j \in X' \subseteq X \Rightarrow x_j \in C_i(X')$ where $X = \bigcup_{j \in M} X_j$. But by assumption, $\exists i \in N \backslash \omega_0$ such that $x_j \in C_i(X_j)$ and $x_{j+1} \in X_j \backslash C_i(X_j)$ for any $x_j \in X$ where $x_1 = x_{m+1}$. So $\forall j \in M \quad |\omega_0| \leq |V(x_{j+1}, X_j; \{C_i(\cdot)\})| < \lceil n/2 \rceil$. Therefore for any $x_j \in X$ to be chosen it must get $\lceil n/2 \rceil - |\omega_0|$ more votes from non-vetoers. Then the rule reduces to a k'-majority rule on $N \backslash \omega_0$, where $k' = \lceil n/2 \rceil - |\omega_0|$. Since k-Majority rule does not satisfy[23] WARP for $1 \leq k < n$ by Theorem **??**, Voting with veto does not also satisfy it. Hence $F \notin \Lambda(WARP)$. □

5.3 Comparative Analysis of SDRs

5.3.1 Normative Conditions for SDRs

11) *Kemeny Rule*

Theorem 58 *Kemeny Rule satisfies Positive and Negative Unanimity. It does not satisfy Monotonicity, Neutrality2, Positive Pareto and NoVeto Power.*

Proof : $(F \in \Lambda^{U^+})$ Let $\forall i \in N \quad (x,y) \in P_i$. Since $P \in \overrightarrow{P}$, $(x,y) \in P$.
$(F \in \Lambda^{U^-})$ Let $\forall i \in N \quad (y,x) \in P_i$. Since $P \in \overrightarrow{P}$, $(x,y) \notin P$.
$(F \notin \Lambda^M)$ First the following Lemma is proved.

Lemma 8 *If* $\forall i,j \in N \backslash \{i_0\} \quad P_i = P_j$ *and* $\forall i \in N \backslash \{i_0\} \quad P_i \neq P_{i_0}$ *then social decision* $P = P_i \in N \backslash \{i_0\}$.

Proof : Assume $\forall i,j \in N \backslash \{i_0\} \quad P_i = P_j$ and $\forall i \in N \backslash \{i_0\} \quad P_i \neq P_{i_0}$. Then $\forall i,j \in N \backslash \{i_0\} \quad d(P_i,P_j) = 0$ hence $\forall i \in N \backslash \{i_0\} \quad \sum_{k \in N} d(P_i,P_k) = d(P_i,P_{i_0}) < \sum_{k \in N} d(P_{i_0},P_k)$ $= (n-1)d(P_i,P_{i_0})$ since $n \geq 3$. Hence $\forall i \in N \backslash \{i_0\} \quad P_i = P$. □

Consider the following example. (Distance matrices of $\overrightarrow{P}$ is given below, the corresponding matrices for $\overrightarrow{P'}$ is not necessary since Lemma **??** will be used).

[23] When $|N|, |\omega_0| > 0$, the condition $k = \lceil n/2 \rceil - |\omega_0| = |N| - |\omega_0|$ implies $\lceil n/2 \rceil = n$, which is not true for $n > 1$, hence the case of unanimity for which WARP is satisfied is irrelevant here.

$\overrightarrow{P}$

P_1	P_2	P_3
w	z	y
x	x	z
y	y	w
z	w	x

$\overrightarrow{P'}$

P'_1	P'_2	P'_3
z	z	y
x	x	z
y	y	w
w	w	x

R_1

–	x	y	z	w
x	–	1	1	0
y	0	–	1	0
z	0	0	–	0
w	1	1	1	–

R_2

–	x	y	z	w
x	–	1	0	1
y	0	–	0	1
z	1	1	–	1
w	0	0	0	–

R_3

–	x	y	z	w
x	–	0	0	0
y	1	–	1	1
z	1	0	–	1
w	1	0	0	–

In this example, $P_3 = P$ since $\sum_{k\in N} d(P_3, P_k) = 14 < \sum_{k\in N} d(P_2, P_k) = 16 < \sum_{k\in N} d(P_1, P_k) = 18$ and $P'_1 = P'_2 = P$, by Lemma **??** since $P'_1 = P'_2$ and $P'_1 \neq P'_3$. Since $V(y, x; \{P_i\}) = V(y, x; \{P'_i\})$ and $V(x, y; \{P_i\}) = V(x, y; \{P'_i\})$, and yPx, but $y\overline{P}'x$, F is not local. Hence $F \notin \Lambda^M$ and $F \notin \Lambda^{Ne_2}$ by Theorem **??**.

($F \notin \Lambda^{PP^+}$) Consider the following example where $A = \{x, y, z, w\}$ and $N = \{1, 2, 3\}$. For simplicity the corresponding distance matrices are added.

P_1	P_2	P_3
x, y	x, y	x
z	z, w	y, z, w
w		

R_1

–	x	y	z	w
x	–	0	1	1
y	0	–	1	1
z	0	0	–	1
w	0	0	0	–

R_2

–	x	y	z	w
x	–	0	1	1
y	0	–	1	1
z	0	0	–	0
w	0	0	0	–

R_3

–	x	y	z	w
x	–	1	1	1
y	0	–	0	0
z	0	0	–	0
w	0	0	0	–

In this example, $\sum_{k\in N} d(P_2, P_k) = 4 < \sum_{k\in N} d(P_1, P_k) = 5 < \sum_{k\in N} d(P_3, P_k) = 7$. So $P_3 \neq P = P_2$. Hence $(x, y) \notin P$.

($F \notin \Lambda^{NVP}$) Consider $\overrightarrow{P}$ in the example that was used above to prove that $F \notin \Lambda^M$. In this example, $P_3 = P$ and hence yPx although $\forall i \in N \backslash \{3\} \quad xP_iy$. □

12) (k_1, k_2)-*majority Rule*

*Remark*19 The analysis of this rule is also sufficient for the analysis of the following rules (because they are special cases as indicated by the following theorems): 13) *Absolute k-majority rules* 14) *Relative k-majority rules.*

First following theorems are introduced. Note that $1 \le k_1 \le n$ and $0 \le k_2 \le n$.

Theorem 59 *(Aleskerov and Vladimirov, 1986) (k_1, k_2)-majority rule is equal to Absolute (resp. Relative) k-majority rule if $k_1 + k_2 = n$ (resp. $k_2 = 0$).*

Theorem 60 *(Aleskerov and Vladimirov, 1986) (k_1, k_2)-majority rule belongs to Symmetrically Central Class.*

Remark 20: By Theorem **??**, Absolute and Relative k-majority rules also belong to Symmetrically Central Class.

Theorem 61 *(k_1, k_2)-majority rule satisfies Positive and Negative unanimity, and Neutrality1. When $1 < k_1 \le n$ it does not satisfy Positive Pareto whereas when $k_1 = 1$ it satisfies Positive Pareto. It satisfies No veto power if and only if $k_1 > 1$ or $(k_1 = 1$ and $k_2 < n - 1)$.*

Proof : $(F \in \Lambda^{U^+})$ Let $\forall i \in N \quad (x, y) \in P_i$. Then $|V(x, y; \{P_i\})| = n \ge k_1$ and $|V(y, x; \{P_i\})| = 0 \le k_2$ since $\overrightarrow{P} \in \mathcal{W}^n$ and hence $(x, y) \in P$.

$(F \in \Lambda^{U^-})$ Let $\forall i \in N \quad (y, x) \in P_i$. Then since $\overrightarrow{P} \in \mathcal{W}^n$, $V(x, y; \{P_i\}) = \emptyset$. Then $|V(x, y; \{P_i\})| = 0 < k_1$ and $(x, y) \notin P$. By Theorem **??**, Absolute and Relative k-majority rules also satisfy Positive and Negative unanimity conditions.

$(k_1 = 1 \Leftrightarrow F \in \Lambda^{PP^+})$ Let $\forall i \in N \quad (y, x) \notin P_i$ and $|V(x, y; \{P_i\})| = 1 \ge k_1$ and $|V(y, x; \{P_i\})| = 0 \le k_2$. Then $(x, y) \in P$. let now $k_1 > 1$. Then since $|V(x, y; \{P_i\})| = 1 < k_1$. Then $(x, y) \notin P$. Then by Theorem **??**, Absolute and Relative k-majority rules also satisfy Positive Pareto condition if and only if $k = 1$.

$(k_1 > 1$ or $(k_1 = 1$ and $k_2 < n - 1) \Leftrightarrow F \in \Lambda^{NVP})$ Let $\forall i \in N \backslash \{j\} \quad (x, y) \in P_i$ and $(y, x) \in P_j$ results $(y, x) \notin P$. Then $|V(y, x; \{P_i\})| = 1 < k_1$ or $|V(y, x; \{P_i\})| = 1 \ge k_1 \ge 1$ but $|V(x, y; \{P_i\})| = n - 1 > k_2$. Let $k_1 > 1$ or $(k_1 = 1$ and $k_2 < n - 1)$ and $\forall i \in N \backslash \{j\} \quad (x, y) \in P_i$ and $(y, x) \in P_j$. Then since $|V(y, x; \{P_i\})| = 1 < k_1$ or $|V(y, x; \{P_i\})| = 1 \ge k_1 \ge 1$ but $|V(x, y; \{P_i\})| = n - 1 > k_2$, $(y, x) \notin P$.

Since (k_1, k_2)-majority rule satisfies No veto power if and only if $k_1 > 1$ or $(k_1 = 1$ and $k_2 < n - 1)$, by Theorem **??** Absolue k-majority rule satisfies NVP if and only if $1 < k \le n$. Note that $(k_1 = 1$ and $k_2 < n - 1) \Rightarrow k_1 + k_2 \ne n$ hence this case does not apply to Absolute k-majority rule. On the other hand, Relative k-majority rule satisfies NVP since the restricition $k_2 = 0 < n - 1$ is always valid for this rule.

□

5.3.2 Rationality Constraints for SDRs

11) *Kemeny Rule*

Theorem 62 *Social decison P by Kemeny rule is a weak order if $\overrightarrow{P} \in \mathcal{W}^n$ and a linear order if $\overrightarrow{P} \in \mathcal{LO}^n$.*

Proof : Omitted since it is obvious by definition. □

12) (k_1, k_2)*-majority rule*

Theorem 63 *Social decison P by (k_1, k_2)-majority rule is irreflexive.*

Proof : Since $\forall x \in A \quad \rceil \exists i \in N$ such that xP_ix, $x\overline{P}x$ since, $F \in \Lambda(PP^-)$ by Theorem **??**.

Theorem 64 *(Aleskerov and Vladimirov, 1986) Social decison P by (k_1, k_2)-majority rule is acyclic if and only if $k_1 > k_2$ and $\lceil k_1/k_2 \rceil \geq |A|$. Whenever $|A| \geq |N|$, social decison P by (k_1, k_2)-majority rule is acyclic if and only if $k_2 = 0$, i.e., the rule is a Relative k-majority rule.*

Corollary 1 *Social decison P by (k_1, k_2)-majority rule is asymmetric if $k_1 > k_2$ and $\lceil k_1/k_2 \rceil \geq |A|$. Whenever $|A| \geq |N|$, social decison P by (k_1, k_2)-majority rule is asymmetric if $k_2 = 0$, i.e., the rule is a Relative k-majority rule.*

Proof : Obvious by Theorem **??** and the fact that Acyclicity implies Asymmetry by Theorem **??**.

Theorem 65 *(Aleskerov and Vladimirov, 1986) Social decison P by (k_1, k_2)-majority rule is a Strict Partial Order if and only if $k_2 = 0$, i.e., the rule is a Relative k-majority rule.*

Corollary 2 *Social decison P by (k_1, k_2)-majority rule is transitive if and only if $k_2 = 0$, i.e., the rule is a Relative k-majority rule.*

Proof : Since by Theorem **??** social decison P by (k_1, k_2)-majority rule is always irreflexive, for P to be a Strict Partial Order, it is necessary and sufficient to be transitive.

Theorem 66 *(Aleskerov and Vladimirov, 1986) Social decison P by (k_1, k_2)-majority rule is never a Weak Order.*

Theorem 67 $\forall \overrightarrow{P} \in \mathcal{W}^n$ *assuming that $0 < k_2 < n$, (cases $k_2 = n$ and $k_2 = 0$ will be investigated seperately below by utilizing Theorem **??**).*

(i) If $k_1 = 1$ then social decison P by (k_1, k_2)-majority rule is not asymmetric and if $k_1 = n$ then it is asymmetric;

(ii) If $1 < k_1 < n$ and $k_2 = 1$ then social decison P by (k_1, k_2)-majority rule is asymmetric;

(iii) If $1 < k_1 < n$ and $1 < k_2 < n$ then social decison P by (k_1, k_2)-majority rule is not asymmetric if $2k_1 \leq |V(x, y; \{P_i\})| + |V(y, x; \{P_i\})| \leq 2k_2$.

Proof : For the case of $k_1 = 1$, consider the following example.

$$\begin{array}{ccc} \underline{1} & \underline{1} & \underline{n-2} \\ \cdot & \cdot & \cdot \\ x & y & x,y \\ y & x & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

In this example, since $|V(x,y;\{P_i\})| = |V(y,x;\{P_i\})| = 1 \geq k_1$ and $|V(x,y;\{P_i\})| = |V(y,x;\{P_i\})| = 1 \leq k_2$, xPy and yPx.

Let $1 < k_1 < n$ and $k_2 = 1$. Let xPy. Then $|V(x,y;\{P_i\})| \geq k_1$ and $|V(y,x;\{P_i\})| \leq 1 < k_1$ then $(y,x) \notin P$.

Let $1 < k_1 < n$ and $1 < k_2 < n$. Let xPy. Then $|V(x,y;\{P_i\})| \geq k_1$ and $|V(y,x;\{P_i\})| \leq k_2$. Let then yPx. Then $|V(y,x;\{P_i\})| \geq k_1$ and $|V(x,y;\{P_i\})| \leq k_2$. Hence $2k_1 \leq |V(x,y;\{P_i\})| + |V(y,x;\{P_i\})| \leq 2k_2$.

Let $k_1 = n$. Then let xPy. Then $y\overline{P}x$ since $\overrightarrow{P} \in \mathcal{W}^n$, $|V(y,x;\{P_i\})| = 0$ and by Theorem **??** $F \in \Lambda^{PP^-}$.

Theorem 68 *(i) Whenever $1 \leq k_1 \leq 2$, if $1 \leq k_2 < n-2$ then social decison P by (k_1,k_2)-majority rule is not connected while $n-1 \leq k_2 \leq n$ it is connected.*

(ii) If $2 < k_1 < n$ then social decison P by (k_1,k_2)-majority rule is not connected.

(iii) If $k_1 = n$ then social decison P by (k_1,k_2)-majority rule is not connected. (The cases of $k_2 = n$ and $k_2 = 0$ will be investigated seperately below by Theorem **??***).*

Proof : For the case of $1 \leq k_1 \leq 2$ and $1 \leq k_2 \leq n-3$, consider the following example.

$$\begin{array}{cc} \underline{k_2+1} & \underline{k_2+1} \\ \cdot & \cdot \\ x & y \\ y & x \\ \cdot & \cdot \end{array}$$

In this example, since $|V(x,y;\{P_i\})| = |V(y,x;\{P_i\})| \geq 2 \geq k_1$ but $|V(y,x;\{P_i\})| > k_2$ and $|V(x,y;\{P_i\})| > k_2$, $x\overline{P}y$ and $y\overline{P}x$.

For the case of $k_1 = 1$ and $n-2 \leq k_2 \leq n$, suppose $\exists x,y \in A$ such that $x \neq y$ and $x\overline{P}y\overline{P}x$ where $\overrightarrow{P} \in \mathcal{LO}$. Then $|V(x,y;\{P_i\})| \geq k_1$ and $|V(y,x;\{P_i\})| \geq k_1$ since $\forall a,b \in A \quad |V(a,b;\{P_i\})| = 0 \Rightarrow bPa$. But then $|V(y,x;\{P_i\})| > k_2 \geq n-2$ and $|V(x,y;\{P_i\})| > k_2 \geq n-2$. Hence $|V(x,y;\{P_i\})| + |V(y,x;\{P_i\})| = n \geq (2n-4)+2$ and therefore $n \leq 2$ which I do not consider.

For the case of $k_1 = 2$ and $n-2 \leq k_2 \leq n$, suppose $\exists x,y \in A$ such that $x \neq y$ and $x\overline{P}y\overline{P}x$ where $\overrightarrow{P} \in \mathcal{LO}$. Let $k_2 = n-2$ then there are four possibilities. (1) $|V(x,y;\{P_i\})| < 2$ and $|V(y,x;\{P_i\})| < 2$ then $|V(x,y;\{P_i\})| \leq 1$ and $|V(y,x;\{P_i\})| \leq 1$ hence $|V(x,y;\{P_i\})| + |V(y,x;\{P_i\})| = n \leq 2$ a case which I do not consider. (2) $|V(x,y;\{P_i\})| < 2$ and $|V(y,x;\{P_i\})| \geq 2$ but $|V(x,y;\{P_i\})| > n-2$. Then $2 > |V(x,y;\{P_i\})| > n-2$ and hence $n = 2$ a case which I do not consider. (3)

Just apply case (2) to the pair (y, x) instead of pair (x, y). (4) Let $|V(x, y; \{P_i\})| \geq 2$ and $|V(y, x; \{P_i\})| > n - 2$ and $|V(y, x; \{P_i\})| \geq 2$ and $|V(x, y; \{P_i\})| > n - 2$. Then $|V(x, y; \{P_i\})| + |V(y, x; \{P_i\})| \geq n + 1 > n$ which is a contradiction. For the cases of $k_2 = n - 1$ and $k_2 = n$ just similar path should be followed as the case of $k_2 = n - 2$.

For the case of $2 < k_1 < n$, consider the following example.

$$\begin{array}{cc} \underline{k_1 - 1} & \underline{k_1 - 1} \\ \cdot & \cdot \\ x & y \\ y & x \\ \cdot & \cdot \end{array}$$

In this example, since $|V(x, y; \{P_i\})| < k_1$ and $|V(y, x; \{P_i\})| < k_1$, $x\overline{P}y$ and $y\overline{P}x$.

For the case of $k_1 = n$, consider the following example where $j > 0$.

$$\begin{array}{cc} \underline{j} & \underline{n - j} \\ \cdot & \cdot \\ x & y \\ y & x \\ \cdot & \cdot \end{array}$$

In this example, $x\overline{P}y$ and $y\overline{P}x$.

Theorem 69 *If* $n = k_1 + 2k_2 + 2$ *then* $F \notin \Lambda(Ngtrv)$.

Proof : Consider the following example.

$$\begin{array}{ccc} \underline{k_1} & \underline{k_2 + 1} & \underline{k_2 + 1} \\ x & x & z \\ z & y & x \\ y & z & y \end{array}$$

In this example, xPy but $x\overline{P}z$ and $z\overline{P}y$. □

13) *Absolute k-majority*

Theorem 70 *Social decison* P *by absolute k-majority rule is irreflexive. For* $1 \leq k < n-1$, *it is not asymmetric, not negatively transitive and not connected. For* $k = n-1$, *it is asymmetric, not negatively transitive and not connected and not acyclic for* $|A| > 2$. *For* $k = 1$, *it is not asymmetric, but negatively transitive and connected. For* $k = n$ *it is a strict partial order.*

Proof : $(F \in \Lambda(Irref))$ By Theorems **??** and **??** $F \in \Lambda(Irref)$.

$(1 \leq k < n - 1 \Rightarrow F \notin \Lambda(Asym))$ Consider the following example where $A = \{x, y, z\}$.

$\underline{k}$	$\underline{k}$	$\underline{n-2k}$
x	y	x, y, z
y, z	x	
	z	

In this example, xPy and yPx violating asymmetry and hence $F \notin \Lambda(Acyc)$, and by $F \in \Lambda(Irref)$ and $F \notin \Lambda(Trv)$ by Theorem **??**.

($k = n-1 \Rightarrow F \in \Lambda(Asym)$) Let xPy. Then $|V(x, y; \{P_i\})| \geq n-1$, but since $\forall \overrightarrow{P} \in \mathcal{W}^n \Rightarrow (\forall i \in N \quad xP_iy \Rightarrow y\overline{P}_ix)$, $|V(y, x; \{P_i\})| \leq 1 < n-1 = k$ since $n \geq 3$. Hence $y\overline{P}x$.

($k = n-1 \Rightarrow F \notin \Lambda(Acyc)$) Consider the following example and let $k = n-1 = 2$.

$\underline{1}$	$\underline{1}$	$\underline{1}$
x	y	z
y	z	x
z	x	y

In this example, $xPyPzPx$.

($k = n \Rightarrow F \in \Lambda(Trv)$) Let xPy and yPz. Then $\forall i \in N \quad xP_iy$ and yP_iz. Since $\overrightarrow{P} \in \mathcal{W}^n$, xP_iz and then xPz hence $F \in \Lambda(Trv)$. And since additionally $F \in \Lambda(Irref)$), $F \in \Lambda(Acyc)$ and hence $F \in \Lambda(Asym)$ by Theorem **??**.

($k = 1 \Rightarrow F \in \Lambda(NegTrv)$) Let xPy and $z \in A$. Then $\exists i \in N \quad$ such that xP_iy. Since $\overrightarrow{P} \in \mathcal{W}^n$, xP_iz or zP_iy and hence xPz or zPy.

($1 < k \leq n \Rightarrow F \notin \Lambda(NegTrv)$) Consider the following example where $A = \{x, y, z\}$.

$\underline{k-2}$	$\underline{1}$	$\underline{1}$	$\underline{n-k}$
x	y	y	x, y, z
y	x, z	z	
z		x	

In this example, $P = \{(y, z)\}$ violating negative transitivity.

($k = 1 \Rightarrow F \in \Lambda(Conn)$) Let $x \neq y$ and $x\overline{P}y$ and $y\overline{P}x$. Then $\forall i \in N \quad x\overline{P}_iy$ and $y\overline{P}_ix$ which is impossible since $\forall i \in N \quad P_i \in \mathcal{LO}$ is assumed.

($1 < k \leq n \Rightarrow F \notin \Lambda(Conn)$) Consider the following example where $A = \{x, y, z\}$.

$\underline{k-2}$	$\underline{1}$	$\underline{1}$	$\underline{n-k}$
x	y	x	x, y, z
y	x	y	
z	z	z	

In this example, $x\overline{P}y$ and $y\overline{P}x$ violating connectedness. □

14) *Relative k-majority*

Theorem 71 *(Aleskerov and Vladimirov, 1986) Social decision P by relative k-majority is a strict partial order.*

Proof : $(F \in \Lambda(Irref))$ By Theorems **??** and **??** $F \in \Lambda(Irref)$.

$(F \in \Lambda(Trv))$ Let xPy and yPz, and let $|V(x,y;\{P_i\})| \geq k$. Consider $i_0 \in V(x,y;\{P_i\})$ and $z \in A$. Since yPz and $\overrightarrow{P} \in \mathcal{W}^n$, either $yP_{i_0}z$ or ($y\overline{P}_{i_0}z$ and $z\overline{P}_{i_0}y$), by definition of the rule. Let $yP_{i_0}z$. Then $xP_{i_0}z$ by transitivity of P_{i_0}. Let, on the other hand, $y\overline{P}_{i_0}z$ and $z\overline{P}_{i_0}y$. Since $P_{i_0} \in \mathcal{W}$ and $xP_{i_0}y$, I have $xP_{i_0}z$ or $zP_{i_0}y$, and hence $xP_{i_0}z$. Then $V(x,y;\{P_i\}) \subseteq V(x,z;\{P_i\})$ and so $|V(x,z;\{P_i\})| \geq |V(x,y;\{P_i\})| \geq k$.

Now consider $j_0 = \{i \in N : x\overline{P}_iy \text{ and } y\overline{P}_ix\} = \omega_1$. There are two possibilities. 1) $xP_{j_0}z$ or $[x\overline{P}_{j_0}z$ and $z\overline{P}_{j_0}x]$, 2) $zP_{j_0}x$. Consider the latter, i.e., $zP_{j_0}x$. Since $P_{j_0} \in \mathcal{W}$, and $x\overline{P}_{j_0}y$ and $y\overline{P}_{j_0}x$, $zP_{j_0}y$ which contradicts yPz, hence the first possibility is relevant. But since $xP_{j_0}z$ or $[x\overline{P}_{j_0}z$ and $z\overline{P}_{j_0}x]$, and $|V(x,z;\{P_i\})| \geq |V(x,y;\{P_i\})| \geq k$, xPz.

Since P is a strict partial order, $F \in \Lambda(Acyc)$ and hence $F \in \Lambda(Asym)$ by Theorem **??**.

$(F \notin \Lambda(NegTrv))$ For $1 \leq k < n$ consider the following example.

$n-k$	k
x,y	z
z	x
	y

In this example, $P = \{(x,y)\}$ violating negative transitivity. For $k = n$ consider the following example.

1	1	1
x	x	z
z	y	x
y	z	y

In this example, xPy but $x\overline{P}z$ and $z\overline{P}y$.

$(F \notin \Lambda(Conn))$ Consider the following example where $A = \{x,y,z\}$.

$k-1$	1	$n-k$
x	y	x,y,z
y	x	
z	z	

For the case of $1 < k \leq n$, in this example, $x\overline{P}y$ and $y\overline{P}x$ violating connectedness. For the case of $k = 1$ consider the following example.

1	1	$n-2$
x	y	x,y,z
y	x	
z	z	

In this example, $x\overline{P}y$ and $y\overline{P}x$ violating connectedness. □

6 Conclusions

First the results are given in tables then comments on these results are made. In tables below (+) represents that the rule belongs to the class and (-) that it does not. If there is an additional assumption for a given result, this will be stated via footnotes. Since all the rules considered satisfy Neutrality1 it will not be included in the tables.

6.1 Social Choice Correspondences

6.1.1 Normative conditions for Coalitional Pareto Rules

In the first column of this table, classes of rules seperated by normative conditions and in the first row the coalitional Pareto rules are listed. As defined before Symmetrically Central Class Λ^{SC} are class of (local) rules that satisfy Positive and Negative non-imposedness, Monotonicity, Neutrality2 and Anonymity.

A plus sign (+) means that the rule satisfies the given condition whereas a minus sign (-) denotes the opposite. Because a rule might satisfy for some parameters and not for others we use the following 2-row 3-column matrix convention in the following manner. Given a rule and a condition, if for parameters $k = 1$ and $q = 0$ a rule satisfies the given condition, the upper left corner of the inner matrix will have a plus sign (+). As the parameters change the signs can change. If a rule satisfies (resp. violates) a condition for all the parameters then there will be a plus (resp. minus) sign. It is written below which position belongs to which parameter values.

$k = 1$ and $q = 0$	$1 < k < n$ and $q = 0$	$k = n$ and $q = 0$
$k = 1$ and $q > 0$	$1 < k < n$ and $q > 0$	$k = n$ and $q > 0$

$NCs \backslash CPRs$	F_s	F_{ss}	F_w
Λ^{SC}	$+$	$+$	$+$
Λ^{U}	$+$	$+$	$+$
Λ^{ND^+}	$+ \ - \ -$ $+ \ - \ -$	$- \ - \ +$ $- \ - \ +$	$+$
Λ^{ND^-}	$+ \ + \ +$ $- \ - \ -$	$+ \ - \ -$ $- \ - \ -$	$+ \ - \ -$ $- \ - \ -$
Λ^{NVP_1}	$- \ + \ +$ $- \ - \ -$	$+ \ + \ -$ $- \ - \ -$	$-$
Λ^{NVP_2}	$+ \ + \ +$ $+ \ - \ -$	$+ \ + \ +$ $- \ - \ +$	$+$
Λ^{RA}	$- \ - \ +$ $- \ - \ +$	$+ \ - \ -$ $+ \ - \ -$	$-$
Λ^{PA}	$+ \ + \ -$ $+ \ + \ -$	$-$	$+$

6.1.2 Rationality constraints for Coalitional Pareto Rules

Since $\Lambda(NE) = \Lambda(FP)$ for all the CPRs by Theorem , $\Lambda(FP)$ will not be listed in the table. For expository purposes $\Lambda(WNR)$ has been used in the table below instead of $\Lambda(ACA^+ \cap WNR, WNR)$.

$RCs \setminus CPRs$	F_s			F_{ss}			F_w
$\Lambda(NE)$	+	−	−	−	−	+	+
	+	−	−	−	−	+	
$\Lambda(WNR)$	−	−	+	+	−	−	−
	−	−	−	−	−	−	
$\Lambda(WARP)$	−	−	+	+	−	−	−
	−	−	−	−	−	−	
$\Lambda(H)$		+			+		+
$\Lambda(C)$	−	−	+	+	+	−	−
	−	−	−	−	−	−	
$\Lambda(O)$	+	−	−	−	−	+	+
	+	−	−	−	−	+	
$\Lambda(ACA)$		−			−		−
$\Lambda(Con^-)$		+			+		+
$\Lambda(Con^+)$	−	−	+	+	+	−	−
	−	−	−	−	−	−	
$\Lambda(PI)$	+	−	−	−	−	+	+
	+	−	−	−	−	+	
$\Lambda(ACA^*)$		−			−		−
$\Lambda(H^-)$		−			−		−

6.1.3 Normative conditions for Positional SCCs

Not to make the table complicated, the following (trivial) facts will not be shown via table: Both plurality and Borda satisfy Positive and Negative non-imposedness, Anonymity, Neutrality1. Since none of these rules satisfy Locality condition, they do not satisfy Monotonicity and Neutrality2 conditions by Theorem **??**.

$NCs \setminus PSCCs$	*Plurality*	*InversePlurality*	*Borda*	*InverseBorda*
Λ^L	−	−	−	−
Λ^U	+	+	+	+
Λ^{ND^+}	−	−	−	−
Λ^{ND^-}	+	−	−	−
Λ^{NVP_1}	+	−	−	−
Λ^{NVP_2}	+	−	−	−
Λ^{RA}	+	+	+	−
Λ^{PA}	+	+	+	−

6.1.4 Rationality constraints for Positional SCCs

For expository purposes $\Lambda(WNR)$ has been used in the table below instead of $\Lambda(ACA^+ \cap WNR, WNR)$. Similarly, $\Lambda(ACA^*)$ and $\Lambda(H^-)$ has been used instead of $\Lambda(ACA^*, ACA^*)$ and $\Lambda(ACA^*, H^-)$.

$RCs \backslash PSCCs$	*Plurality*	*InversePlurality*	*Borda*	*InverseBorda*
$\Lambda(NE)$	$+$	$+$	$+$	$+$
$\Lambda(FP)$	$-$	$-$	$-$	$-$
$\Lambda(WNR)$	$-$	$-$	$-$	$-$
$\Lambda(WARP)$	$-$	$-$	$-$	$-$
$\Lambda(H)$	$-$	$-$	$-$	$-$
$\Lambda(C)$	$-$	$-$	$-$	$-$
$\Lambda(O)$	$-$	$-$	$-$	$-$
$\Lambda(ACA)$	$-$	$-$	$-$	$-$
$\Lambda(Con^-)$	$-$	$-$	$-$	$-$
$\Lambda(Con^+)$	$-$	$-$	$-$	$+$
$\Lambda(PI)$	$-$	$-$	$-$	$-$
$\Lambda(ACA^*)$	$-$	$-$	$-$	$-$
$\Lambda(H^-)$	$-$	$-$	$-$	$-$

6.2 Functional Voting Rules

6.2.1 Normative conditions for Functional Voting Rules

As defined before Symmetrically Central Class Λ^{SC} are class of (local) rules that satisfy Positive and Negative non-imposedness, Monotonicity, Neutrality2 and Anonymity. Below AV denotes Approval Voting, k-M denotes k-majority rules and VwV denotes Voting with Veto. Not to make the table complicated, the following facts will not be shown via table: All the FVRs satisfy Positive and Negative non-imposedness, and Neutrality1. Approval Voting does not satisfy Locality hence Monotonicity and Neutrality2. All the FVRs but Voting with Veto does not satisfy Anonymity hence although it satisfies Monotonicity and Neutrality2 (hence Locality) it is not in Λ^{SC}.

$NCs \backslash FVRs$	*AV*	k-M	VwV
Λ^{SC}	$-$	$+$	$-$
Λ^{U^+}	$+$	$+$	$+$
Λ^{U^-}	$+$	$+$	$+$
Λ^{NVP}	$+$	$-$ $+$ $+$	$+$
Λ^{RA}	$+$	$-$ $-$ $+$	$+$

6.2.2 Rationality constraints for Functional Voting Rules

For expository purposes $\Lambda(WNR)$ has been used in the table below instead of $\Lambda(ACA^+ \cap WNR, WNR)$. Similarly, $\Lambda(ACA^*)$ and $\Lambda(H^-)$ has been used instead of $\Lambda(ACA^*, ACA^*)$ and $\Lambda(ACA^*, H^-)$.

$RCs\backslash FVRs$	AV	k-M	VwV
$\Lambda(NE)$	+	−	−
$\Lambda(FP)$	−	−	−
$\Lambda(WNR)$	−	−	+
$\Lambda(WARP)$	−	− − +	−
$\Lambda(H)$	−	+	+
$\Lambda(C)$	−	− − +	−
$\Lambda(O)$	−	+ − −	−
$\Lambda(ACA)$	−	−	−
$\Lambda(Con^-)$	−	+	+
$\Lambda(Con^+)$	−	− − +	−
$\Lambda(PI)$	−	+ − −	−
$\Lambda(ACA^*)$	−	+	+
$\Lambda(H^-)$	−	+	+

6.3 Social Decision Rules

6.3.1 Normative conditions for Social Decision Rules

As defined before Symmetrically Central Class Λ^{SC} are class of (quasilocal) rules that satisfy Positive and Negative non-imposedness, Monotonicity, Neutrality2, Anonymity and Negative Pareto. Below K-r denotes Kemeny rule, $(k_{\acute{1}}, k_2)$-M denotes $(k_{\acute{1}}, k_2)$-majority rules and *Abs-k* (resp.*Rel-k*) denotes Absolute (resp. Relative) k-majority rules.

Not to make the table complicated, the following facts will not be shown via table: All the SDRs satisfy Positive and Negative non-imposedness, Neutrality1, Negative Pareto and Anonymity. Kemeny rule does not satisfy Quasilocality hence Monotonicity and Neutrality2. All the FVRs but Voting with Veto does not satisfy Anonymity hence although it satisfies Monotonicity and Neutrality2 (hence Locality) it is not in Λ^{SC}.

$NCs\backslash FVRs$	*K-r*	$(k_{\acute{1}}, k_2)$-M	*Abs-k*	*Rel-k*
Λ^{SC}	−	+	+	+
Λ^{U^+}	+	+	+	+
Λ^{U^-}	+	+	+	+
Λ^{PP^+}	−	+ − −	+ − −	+ − −
Λ^{NVP}	−	− + +	− + +	+

6.3.2 Rationality constraints for Social Decision Rules

For expository purposes $\Lambda(WNR)$ has been used in the table below instead of $\Lambda(ACA^+ \cap WNR, WNR)$. Similarly, $\Lambda(ACA^*)$ and $\Lambda(H^-)$ has been used instead of $\Lambda(ACA^*, ACA^*)$ and $\Lambda(ACA^*, H^-)$.

$RCs \backslash SDRs$	*K-r*	(k_1, k_2)-M	*Abs-k*	*Rel-k*
$\Lambda(Asym)$	$+$	$-$	$-$	$+$
$\Lambda(Acyc)$	$+$	$-$	$-$	$+$
$\Lambda(Trv)$	$+$	$-$	$-$	$+$
$\Lambda(Ngtrv)$	$+$	$-$	$-$	$-$
$\Lambda(Conn)$	$+$	$-$	$-$	$-$

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A Appendix

A.1 Social Choice Correspondences (SCCs)

In this section, the individual opinions are represented as weak orders and the social decision is constructed as a choice function.

The procedures are classified into six[24]:

1. SCCs using majority relation,
2. SCCs using a utility function
3. SCCs using tournament matrix,
4. Positional SCCs,
5. Coalitional Pareto Rules
6. SCCs using Lexicographic utilities.

Unless otherwise stated $\overrightarrow{P}$ is a profile of linear orders, the given presentation is A and the given coalition is N . Throughout the definitions the expressions are to hold $\forall \overrightarrow{P} \in \mathcal{L}$, $\forall \omega \in \Omega$, $\forall X \in \mathcal{A}$.

A.1.1 Procedures using majority relation μ

1 *Simple Majority rule*

The alternative is chosen if declared best by at least a majority of voters $\lceil n/2 \rceil$, i.e.,

$a \in F(\overrightarrow{P}) \Longleftrightarrow n^+(a, \overrightarrow{P}) \geq \lceil n/2 \rceil$.

2 *Condorcet Winner* (*CW*)

Given N , if an alternative $a \in A$ is preferred to every $b \in A \backslash \{a\}$ by a majority of N then this alternative is called the Condorcet Winner (this majority need not consist the same voters), i.e.

$a \in F(\overrightarrow{P}) \Longleftrightarrow [\forall x \in A \backslash \{a\}, \exists a \text{ such that } a\mu x]$.

Assume that the number of voters in N is odd, then a unique CW will be obtained. In contrast to the winner of the simple majority rule, an alternative may be a CW even if it is declared top by none of the voters. Every simple majority winner is also a CW but not vice versa.

3 *Multi Step Binary Voting* (*MSBV*)

[24] A similar classification can be found in (Moulin, 1987, part 4), which considers the SCCs in mainly two general groups, 1) Condorcet type (in our case the SCCs, which use the majority relation) and 2) Borda type (in our case positional SCCs).

Suppose the alternatives are ordered beforehand. Then starting from the first compare the alternative at hand with the next in terms of majority relation. Eliminate the one which is dominated and do this until one alternative is left, i.e.

Starting from $i = 1$

If $a_i \mu a_{i+1}$ then choose a_i and compare a_i with a_{i+2},which is next in the order;

If $a_{i+1} \mu a_i$ then choose a_{i+1} and compare a_{i+1} with a_{i+2}

and so on until there is no alternative to compare the one at hand. This alternative is chosen.

4 *General Optimal Choice Axiom* (*GOCHA*)

This procedure chooses the undominated alternatives forming a cycle in μ. It is defined by the following binary relation S on A where S is called the relation of indirect domination[25]:

$xSy \Longleftrightarrow \exists$ an order of alternatives in X, $x = x_1, x_2, ..., x_n = y$ such that $x_1 \mu x_2 \mu ... \mu x_n$ and

$\rceil \exists$ an order of alternatives in X, $y = z_1, z_2, ..., z_n = x$ such that

$y = z_1 \mu z_2 \mu ... \mu z_n = x$.

So S is a binary relation which includes (x, y) if and only if x dominates an alternative that dominates another (and possibly many others) which dominates y but y can not do the same. The procedure is to choose undominated alternatives, i.e.

$F(\overrightarrow{P}) = \{a \in A : \rceil \exists x \in A \text{ such that } xSa\}$.

5 *von Neumann-Morgenstern Solution (vN-M Solution)*

A set $Q \subseteq A$ is called *vN-M solution* if no alternative is preferred to another in terms of majority relation μ in Q but for any alternative in $A \backslash Q$ there exists one alternative from Q dominating it in terms of μ . A vN-M solution Q is *minimal* if no subset of Q is a vN-M solution, i.e.,

If $([a \in Q \text{ and } b \in Q] \Rightarrow [a \overline{\mu} b \text{ and } b \overline{\mu} a] \text{ and } \forall c \in A \backslash Q\ , \exists x \in Q \text{ such that } x \mu c)$. Then $F(\overrightarrow{P}) = Q$.

If the vN-M solution is not unique, then social decision is the union of these sets.

6 *Minimal Dominant Set*

A set $Q \subseteq A$ is called *dominant* if each alternative in Q dominates every alternative outside Q in terms of μ. A dominant set Q is *minimal* if no subset of Q is dominant, i.e.,

$Q \subseteq A$ is called dominant if $x \in Q \Rightarrow [\forall y \in A \backslash Q, \quad x \mu y]$.

Q is minimal if $\rceil \exists Q' \subseteq Q$ such that Q' is dominant.

Then $F(\overrightarrow{P}) = Q$. If such sets are not unique, then social decision is the union of these sets.

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[25] In other words, S is the transitive closure of μ. Transitive closure is defined as follows for any binary relation P on set A:

$P^t = P \cup P \circ P \cup P \circ P \circ P.....$

where, $P \circ P = \{(x, z) : (x, y) \in P \text{ and } (y, z) \in P\}$

and $\circ$ is called the composition operation on binary relations

7 *Minimal Undominated Set* (*Schwartz's Procedure*)

A set $Q \subseteq A$ is called *undominated* if no alternative from $A \backslash Q$ dominates any alternative inside Q in terms of μ. An undominted set Q is *minimal* if no subset of Q is undominated; i.e.,

$Q \subseteq A$ is called undominated if $x \in Q \Rightarrow \rceil \exists y \in A \backslash Q$ such that $y \mu x$.

Q is minimal if $\rceil \exists Q' \subset Q$ such that Q' is undominated.

Then $F(\overrightarrow{P}) = Q$. If such sets are not unique, then social decision is the union of these sets.

8 *Minimal Weakly Stable Set* (*Aleskerov's Procedure*)

A set $Q \subseteq A$ is called *weakly stable* if it has the following property: For any alternative $x \in Q$, if there exists an alternative $y \in A \backslash Q$ which dominates this alternative x in terms of μ, then there exists another alternative $z \in Q$, which dominates y. A weakly stable set Q is *minimal* if no subset of Q is weakly stable, i.e.,

$Q \subseteq A$ is called weakly stable if for any $x \in Q$, $\exists y \in A \backslash Q$ such that $y \mu x$, then $\exists z \in Q$ such that $z \mu y$; Q is minimal if $\overline{\exists} Q' \subseteq Q$ such that Q' is weakly stable.

Then $F(\overrightarrow{P}) = Q$. If such sets are not unique, then social decision is the union of these sets.

9 *Fishburn's Rule*

Consider the upper contour sets of all alternatives in A regarding the majority relation, that is $D(x)$ for any $x \in A$. Construct binary relation γ such that $x \gamma y \Longleftrightarrow D(x) \subset D(y)$. Then the undominated alternatives in terms of γ is chosen, i.e.,

$x \in F(\overrightarrow{P}) \Longleftrightarrow [\rceil \exists y \in A$ such that $y \gamma x]$.

10 *Uncovered Sets*

Consider the lower contour sets of all alternatives in A regarding the majority relation, that is $L(x)$ for any $x \in A$. Construct binary relation δ such that $x \delta y \Longleftrightarrow L(x) \supset L(y)$ Then the undominated alternative in terms of δ is chosen, i.e.,

$x \in F(\overrightarrow{P}) \Longleftrightarrow [\rceil \exists y \in A$ such that $y \delta x]$.

11 *Richelson's Rule*

Consider the upper and lower contour sets of all alternatives in A regarding the majority relation that is $D(x)$ and $L(x)$ for any $x \in A$. Constrcuct binary relation σ such that $x \sigma y \Longleftrightarrow [\ L(x) \supseteq L(y)$ and $D(x) \subseteq D(y)$ and (either $[L(x) \supset L(y)\ ]$or $[D(x) \subset D(y)]$)].Then the undominated alternative in terms of σ is chosen, i.e.,

$x \in F(\overrightarrow{P}) \Longleftrightarrow \rceil \exists y \in A$ such that $y \sigma x$.

12 *Dodgson's Procedure*

This procedure chooses the CW if it exists. For the cases where there does not exist a CW, define an *inversion* in $\overrightarrow{P}$ for $x \in A$,which is a change from xP_iy to yP_ix for any $P_i \in \overrightarrow{P}$. The alternative which needs the minimum number of inversions to become a CW is chosen. Given $\overrightarrow{P}$ the number of inversions is denoted as $t(x, \overrightarrow{P})$ for any $x \in A$ .The procedure is as follows:

If a is the CW then $\{a\} = F(\overrightarrow{P})$ else

$F(\overrightarrow{P}) = \{x \in A : \forall y \in A \ , \ t(x, \overrightarrow{P}) \leq t(y, \overrightarrow{P})\}$.

A.1.2 Procedures using a utility function

In each of these procedures it is constructed some intermediate 'utility' function which assigns some numerical values to alternatives. These values represent the desirability of the alternative to society and society decides on the alternatives by choosing the one(s) having the maximum numerical value.

13 *Copeland's Rule I*

Define $u(x)$ as the difference of cardinalities of lower and upper contour sets of x, i.e., $u(x) = |L(x)| - |D(x)|$. Then the alternatives with maximum utility value are chosen, i.e.,

$x \in F(\overrightarrow{P}) \iff [\forall y \in A \ , \ u(x) \geq u(y)]$.

14 *Copeland's Rule II*

Define $u(x)$ as the difference of cardinality of lower contour set of x, i.e., $u(x) = |L(x)|$. Then the alternatives with maximum utility value are chosen, i.e.,

$x \in F(\overrightarrow{P}) \iff [\forall y \in A \ , \ u(x) \geq u(y)]$.

15 *Young's Procedure*

Given N and $\overrightarrow{P}$, if $a \in A$ is a CW, then it is chosen and procedure ends. If not, define partial Condorcet winner as $CW(j)$ as a CW on some coalition ω^* with $|\omega^*| = j < n$. Define $u(x)$ as the cardinality of the greatest coalition where x is the partial CW, i.e., $u(x) = i \Leftrightarrow x = CW(i)$. Then the alternatives with maximum utility value are chosen, i.e.,

$a \in F(\overrightarrow{P})$ if a is $CW(j_0)$ for $j_0 = \max_{1 \leq j \leq n} \{j\}$.

A.1.3 Procedures using tournament matrix

16 *Maxmin Procedure* (*Simpson's Procedure*)

Construct a matrix S^+ such that $\forall a, b \in A, \quad S^+ = \{n(a,b)\}$ with $\{n(a,a)\} = \infty$, where rows and columns correspond to the set of alternatives in A. Then choose row minima from every row (for every alternative). For any $c \in A$, this number (row minimum) shows the worst performance of c against its 'toughest' contestant. Then choose the alternative to which maximum of row minima correspond, that is, choose the alternative which performed best against its contestants, i.e.,

$x \in F(\overrightarrow{P})$ if $n(x,y) = \max_{a \in A}\{\min_{b \in A}\{n(a,b)\}\}$ for some $y \in A$.

17 *Minimax Procedure*

Construct a matrix S^- such that $\forall a, b \in X, \quad S^- = \{n(a,b)\}$, with $n(a,a) = -\infty$, where rows and columns correspond to the set of alternatives in A. Then choose column maxima from every column (for every alternative). For any $c \in A$, this number (column maximum) shows the worst performance of c against its 'toughest' contestant. Then choose the alternative to which minimum of column maxima correspond, that is choose the alternative which performed best against its contestants, i.e.,

$x \in F(\overrightarrow{P})$ if $n(x,y) = \min_{b \in A}\{\max_{a \in A}\{n(a,b)\}\}$ for some $y \in A$.

A.1.4 Positional SCCs

18 *Plurality rule*

The alternative declared the best by a maximal number of voters is chosen, i.e.,

$a \in F(\overrightarrow{P}) \iff [\forall x \in A, \quad n^+(a, \overrightarrow{P}) \geq n^+(x, \overrightarrow{P})]$.

19 *Runoff Procedure*

First simple majority is applied and if as a result $F(\overrightarrow{P}) \neq \emptyset$ Runoff procedure ends. Otherwise eliminate every alternative but two, those having the maximal number of top votes. Then contract $\overrightarrow{P}$ to those two alternatives and apply the simple majority rule to them, i.e.,

If $\exists x \in A$ such that $n^+(x, \overrightarrow{P}) \geq \lceil n/2 \rceil$, then $x \in F(A, \overrightarrow{P})$ and procedure is over.
If not then first determine $b \in A$ with the highest number of top votes, i.e.,
$\forall y \in A, \quad n^+(b, \overrightarrow{P}) \geq n^+(y, \overrightarrow{P})$,
then determine $c \in A \backslash \{b\}$ with the next highest number of top votes in A, i.e.,
$\forall z \in A/\{b\}, \quad n^+(c, \overrightarrow{P}) \geq n^+(z, \overrightarrow{P})$.
Then apply simple majority to $X = \{b, c\}$ under $\overrightarrow{P}/X$.

20 *Hare System*

First simple majority is applied and if as a result $F(\overrightarrow{P}) \neq \emptyset$ Hare procedure ends. If as a result $F(\overrightarrow{P}) = \emptyset$, then one alternative is eliminated from A, the one who got a minimal number of top votes. Then simple majority is reapplied to the remaining set and procedure continues similarly, i.e.,

If $\exists x \in A$ such that $n^+(x, \overrightarrow{P}) \geq \lceil n/2 \rceil$ then $x \in F(\overrightarrow{P})$
If not eliminate $c \in A$ such that $\forall b \in A, \ n^+(c, \overrightarrow{P}) \leq n^+(b, \overrightarrow{P})$
and apply the same procedure to $X = A \backslash \{c\}$ and $\overrightarrow{P}/X$.

If there does not exist a simple majority winner then, continue with the procedure by contracting the set in consideration until reaching a social decision.

21 *Inverse Plurality rule*

The alternative declared worst by a minimal number of voters is chosen, i.e.,
$a \in F(\overrightarrow{P}) \iff [\forall x \in A, \quad n^-(a,\overrightarrow{P}) \le n^-(x,\overrightarrow{P})]$.

22 *Borda Procedure*

Given a profile of linear orders $\overrightarrow{P}$, consider $x \in A$ and assign to x a score $r_i(x,\overrightarrow{P})$, which is the cardinality of lower contour set under $P_i \in \overrightarrow{P}$, i.e., $r_i(x,\overrightarrow{P}) = |L_i(x)| = |\{b \in A : aP_ib\}|$. The sum of these scores through every $i \in N$ is called the Borda Count of the alternative. Then choose the one who has highest Borda Count, i.e.,

Let $r(a,\overrightarrow{P}) = \sum_{i=1}^{n} r_i(a,P_i)$ where $r(a,\overrightarrow{P})$ is the Borda Count of a . Then $a \in F(A,\overrightarrow{P}) \iff [\forall b \in A, \quad r(a,\overrightarrow{P},A) \ge r(b,\overrightarrow{P},A)]$.

23 *Black's Procedure*

This procedure is a combination of two previously defined SCPs: If there exists a CW, then it is chosen else apply the Borda Procedure.

24 *Inverse Borda Procedure*

Compute the Borda Count of each alternative as defined in Borda Procedure. Then eliminate the one who has lowest Borda Count and contract $\overrightarrow{P}$ to the remaininig set and compute the new Borda Scores in the contracted profile. Then eliminate another one similarly and go on like this until there is no alternative to eliminate from the contracted set, i.e.,

Let $r(a,\overrightarrow{P})$ be the Borda Count of any $a \in A$ under $\overrightarrow{P}$.

Then eliminate $c \in A$ where $\forall x \in A, \quad r(c,\overrightarrow{P}) \le r(x,\overrightarrow{P})$.

and apply the same procedure to $X = A\backslash\{c\}$ and $\overrightarrow{P}/X$.

Continue with the procedure by contracting the set in consideration until reaching a social decision.

25 *Nanson's Procedure*

Given $\overrightarrow{P}$compute the Borda Counts of each alternative in A. Then compute the average of these counts. Then eliminate those, who have lower scores than the average value. Then compute the new Borda Scores from the contracted profile. Then eliminate another one similarly and go on like this until there is no alternative to eliminate from the contracted set, i.e.,

Compute $\overline{r} = \left(\sum_{a\in A} r(a,\overrightarrow{P})\right) / |A|$. Then eliminate $c \in A$ if $r(c,\overrightarrow{P}) < \overline{r}$ and construct $X = \{a \in A : r(a,\overrightarrow{P}) \ge \overline{r}\}$.

Then apply the same procedure to X and$\overrightarrow{P}/X$.

Continue with the procedure by contracting the set in consideration until reaching a social decision.

26 *Coomb's Procedure*

Given A and $\overrightarrow{P}$, a profile of linear orders, the alternative declared worst by a maximal number of voters is eliminated. Then contract $\overrightarrow{P}$ to the remaining set X. Then eliminate another one similarly and go on like this until there is no alternative to eliminate from the contracted set, i.e.,

Given A and $\overrightarrow{P}$, eliminate $c \in A$ such that $\forall x \in A, \quad n^{-}(c, \overrightarrow{P}) \leq n^{-}(x, \overrightarrow{P})$. Then apply the same procedure to $X = A \backslash \{c\}$ and $\overrightarrow{P}/X$.

Continue with the procedure by contracting the set in consideration until reaching a social decision.

A.1.5 Coalitional Pareto rules

27 *Strong k−majoritarian q−Pareto rule*

Let $f(X, \overrightarrow{P}; \{i\}, q)$ be defined as q top elements in preference of i when contracted to X. Let $\mathcal{I} = \{I \subseteq N : |I| = k\}$ where $1 \leq k \leq n$. The rule chooses an alternative if it is among the top q elements in the preference of every agent in some I with cardinality k, i.e.,

$C(X) = \bigcup_{I \in \mathcal{I}} \bigcap_{i \in I} f(X, \overrightarrow{P}; \{i\}, q)$

where $f(X, \overrightarrow{P}; \{i\}, q) = \{x \in X : card\ D_i(x) \leq q\}$.

28 *Weak k−majoritarian q−Pareto rule*

Let $f(X, \overrightarrow{P}; I, q)$ be defined as elements in profile $\overrightarrow{P}$ contracted to I, having upper contour sets having an intersection with cardinality smaller than q. Let $\mathcal{I} = \{I \subseteq N : |I| = k\}$ where $1 \leq k \leq n$. The rule chooses an alternative if it has such upper contour sets with regard to one coalition I with cardinality k, i.e.,

$C(X) = \bigcup_{I \in \mathcal{I}} f(X, \overrightarrow{P}; I, q)$

where $f(X, \overrightarrow{P}; I, q) = \{x \in X : \forall i \in N \quad |X \cap D_i(x)| \leq q\}$.

29 Strongest *k−majoritarian q−Pareto rule*

Let $f(X, \overrightarrow{P}; I, q)$ be defined as elements in profile $\overrightarrow{P}$ contracted to I, having upper contour sets having an intersection with cardinality smaller than or equal to q. (Here q can be thought as a performance criterion for the alternatives[26]). Let $\mathcal{I} = \{I \subseteq N : |I| = k\}$ where $1 \leq k \leq n$. The rule chooses an alternative if it has such upper contour sets with regard to all coalitions with cardinality k, i.e.,

$C(X) = \bigcap_{I \in \mathcal{I}} f(X, \overrightarrow{P}; I, q)$

where $f(X, \overrightarrow{P}; I, q) = \{x \in X : \forall i \in N \quad |X \cap D_i(x)| \leq q\}$.

[26]For the present model, it will be assumed that $0 \leq q \leq |A| - 1$. This is because $q < 0$ implies that $\forall X \quad C(X) = \emptyset$ and $q \geq |A|$ implies that $\forall X \quad C(X) = X$.

A.1.6 Procedures using Lexicographic utility functions

30 *Leximin Rule*

Suppose for each $x \in A$ there is a vector of utility values and the components of the vector are values assigned by individuals to that alternative, i.e., given N , $u : A \longrightarrow R^n$. Let $u(x)$ show the values assigned to x by voters in N and with the following property: $u(x) = [u_1(x), u_2(x), ..., u_n(x)]$ where $u_i(x)$, $i \in N$ are in non-decreasing order, i.e. $u_1(x) \leq u_2(x) \leq ..... \leq u_n(x)$. Then it is said that x is lexicographically preferred to y, i.e. $x(lp)y$, iff $[\exists i \in N, \quad u_i(x) > u_i(y)$ and $\forall j \in N \setminus \{i\}, \quad u_i(x) = u_i(y)]$. Then undominated alternative is chosen, i.e. $F(A; u(\cdot)) = \{z \in A : \rceil \exists y \in A$ such that $y(lp)z \}$.

A.2 Social Decision Rules (SDRs)

In this section, the individual opinions are in the form of binary relations and the social decision is also in the form of a binary relation.

31 *Kemeny Rule*

Given $\overrightarrow{P}$ construct matrix $\forall a, b \in A$, $R = \{r^i_{ab}\}$where $r^i_{ab} = 1$ if aP_ib , $r^i_{ab} = 0$ otherwise. Let the distance $d(P_i, P_j)$ of a linear order to another linear order be defined as follows: $\forall a, b \in A$, $\forall i, j \in N$, $d(P_i, P_j) = \sum\limits_{a,b} \left| r^i_{ab} - r^j_{ab} \right|$. The distance $d(P_i, \overrightarrow{P})$ of a linear order P_i to the profile $\overrightarrow{P}$, is defined as follows: $d(P_i, \overrightarrow{P}) = \sum\limits_{j \in N} d(P_i, P_j)$. Finally, the linear order with minimum distance to its profile is the social decision from that profile, i.e.

$F(\overrightarrow{P}) = P_k \in \overrightarrow{P}$, where $\forall P_j \in \overrightarrow{P}$, $d(P_k, \overrightarrow{P}) \leq d(P_j, \overrightarrow{P})$.

32 *τ-system of* (k_1,k_2) *-majorities*

Given a profile of weak orders $\overrightarrow{P}$ and N , in (k_1,k_2) *-majority procedure* the pair $(x, y) \in A \times A$ is included in social decision P if $n_1 \geq k_1$ voters include this pair in their preferences, $n_2 \leq k_2$ voters include $(y, x) \in A \times A$ in their preferences and the rest of the voters ($n - (n_1 + n_2)$ abstain to include (x, y) or (y, x) in their preferences) , i.e.,

Given k_1 and k_2 ,

$$F(\overrightarrow{P}) = P = \{(x, y) \in A \times A : n_1 \geq k_1 \text{ and } n_2 \leq k_2\},$$

where $card\{i \in N : (x, y) \in P_i\} = n_1$ and $card\{j \in N : (y, x) \in P_j\} = n_2$ and $n_1 + n_2 \leq n$.

This rule can be generalized to *τ-system of* (k_1,k_2) *-majorities.* A procedure is *τ-system of* (k_1,k_2) *-majorities* if it is a union of the procedures (k_1,k_2)*-majorities.* For example, suppose $\tau = 3$, then an admissible procedure is $\{(6, 3), (4, 2), (3, 1)\}$-majority, i.e. the decision is made if at least 6 voters vote for it and not more than 3 against it; or at least 4 voters vote for it and not more than 2 against it;or at least 4 voters vote for it and not more than 2 against it.

33 *Absolute k -majority*

Given a profile of weak orders $\overrightarrow{P}$ and N, in this procedure the pair $(x, y) \in A \times A$ is included in social decision P if $n_1 \geq k$ voters include this pair in their preferences independent of other preferences, i.e.,

Given k,

$$F(\overrightarrow{P}) = P = \{(x, y) \in A \times A : n_1 \geq k\},$$

where $card\{i \in N : (x, y) \in P_i\} = n_1$.

34 *Relative k -majority*

Given a profile of weak orders $\overrightarrow{P}$ and N, in this procedure the pair $(x, y) \in A \times A$ is included in social decision P if $n_1 \geq k$ voters include this pair in their preferences and other voters abstain to include (x, y) or (y, x) in their preferences, i.e.,

Given k_1,

$$F(\overrightarrow{P}) = \{(x, y) : [card\{i : (x, y) \in P_i\} = n_1 \geq k_1] \text{ and}$$

$$card\{j : (y, x) \notin P_j \text{ and } (x, y) \notin P_j\} = n - n_1\}.$$

A.3 Functional Voting Rules (FVRs)

In this section, the individual opinions are in the form of choice functions and the social decision is also in the form of a choice function.

Unless otherwise stated the given presentation is A and the given coalition is N.

Throughout the definitions the expressions are to hold $\forall \overrightarrow{C} \in \mathcal{C}^N$, $\forall \omega \in \Omega$, $\forall X \in \mathcal{A}$.

35 *Approval Voting*[27]

Given $\overrightarrow{C}(\cdot)$ every $i \in N$ chooses $C_i(A)$ from A where $C_i(A) \subseteq A$. Then for each alternative $x \in A$, the number of voters who choose x from A is computed. The alternative with greatest such number is chosen if it is chosen at least by one voter, i.e.

$C(A) = F(\overrightarrow{C}) = \{x \in \bigcup_{i \in N} C_i(A) : \forall y \in X,$
$card\{i \in N : x \in C_i(A)\} \geq card\{i \in N : y \in C_i(A)\}\}.$

36 *k $-$Majority rules*

36.1 *One vote for* $(k = 1)$

[27]This version of Approval Voting is by (Sertel, 1988). Here, an alternative is chosen as social decision, only if it is chosen by at least one of the voters whereas in Fishburn and Brams' version whole presentation is chosen if each of the voters declare empty choice.

Given $\overrightarrow{C}(\cdot)$ every $i \in N$ chooses $C_i(A)$ from A where $C_i(A) \subseteq A$. For each alternative the number of voters who indicated that alternative in his/her choice set is computed. The alternative is in the social decision $C(A)$ if there exists at least one voter who includes it in his/her choice set, i.e.,

$C(A) = F(\vec{C}) = \{x \in A : card\,\{i \in N : x \in C_i(A)\} \geq 1\}$.

36.2 *k− Majority rules* $(2 \leq k \leq n-1)$

Given $\overrightarrow{C}(\cdot)$ every $i \in N$ chooses $C_i(A)$ from A where $C_i(A) \subseteq A$. For each alternative the number of voters who indicated that alternative in his/her choice set is computed. The alternative is in the social decision $C(A)$ if there exists at least k voters who includes it in his/her choice set where $2 \leq k \leq n-1$, i.e.,

$C(A) = F(\vec{C}) = \{x \in A : card\,\{i \in N : x \in C_i(A)\} \geq k\}$ where $2 \leq k \leq n-1$.

36.3 *Unanimity* $(k = n)$

Given $\overrightarrow{C}(\cdot)$ every $i \in N$ chooses $C_i(A)$ from A where $C_i(A) \subseteq A$. For each alternative the number of voters who indicated that alternative in his/her choice set is computed. The alternative is in the social decision $C(A)$ if all the voters include it in his/her choice set, i.e.,

$C(A) = F(\vec{C}) = \{x \in A : card\,\{i \in N : x \in C_i(A)\} = n\}$.

37 *Voting with Veto*

In this procedure, $A = \{x\}$ is the only admissible presentation. As a social decision, choice of x represents acception of a "proposal" x, and choice of empty set means preserving "status quo".

The set of voters is partitioned into two: the vetoers and others. The set N_1 is called the set of vetoers where $\forall j \in N_1$, $C_j(A) = \emptyset \Rightarrow C(A) = \emptyset$. So if a voter who has the right to veto chooses empty set from X, then the social decision is empty set. Otherwise, to get $\{x\}$ as social decision there must be a simple majority of voters choosing x. If there does not exist simple majority the decision is again empty set, i.e.

Let $N = N_1 \cup N_2$ and $N_1 \cap N_2 = \emptyset$, where N_1 is the set of vetoers.

If ($\exists i \in N_1$ such that $C_i(A) = \emptyset$) or ($\forall j \in N_1$, $C_j(A) = x$ but $card\{k \in N : C_k(\cdot) = x\} < \lceil n/2 \rceil$) then $C(A) = F(\vec{C}) = \emptyset$. Else $C(A) = \{x\}$.